Density laws

Linear density function

We consider a radially symmetric density model of the form

$$\rho(r) = \rho_0 - br,$$

for radii $0 \le r \le R_E$, where ρ_0 is the density in the Earth's center, b is a positive constant, and R_E is the radius of the Earth.

Moreover, we assume that $\rho(R_E) = 2700 \ kg/m^3$.

Since the mean density is known to be $\rho_M = 5515 \ kg/m^3$, the density in the center of the Earth, ρ_0, must be larger than ρ_M.

Mass of the Earth

The radius of the Earth is $R_E = 6.371 \times 10^6 \ m$, its mass can therefore be evaluated at

$$m_E = \rho_M \frac{4\pi R_E^3}{3}.$$

With our linear density model $\rho(r)$ we obtain for the total mass of a sphere with radius R_E

$$m_E = 4\pi \int_0^{R_E} (\rho_0 - br) r^2 dr.$$

However, this equation is not sufficient to evaluate the two constants ρ_0 and b.

The necessary second equation is

$$\rho(R_E) = \rho_0 - bR_E \ .$$

Implementation

In the remaining part of this notebook, we will use MATLAB's Symbolic Math Toolbox to first evaluate the integral and the related constants. Later, we derive the gravitational attraction inside the Earth.

To this end, we define symbolic variables:

```
syms rho_0 rho_E b R_E rho_M m m_E r real
```

The mass of the Earth can be evaluated simply by multiplying its mean density ρ_M with its spherical volume. We store this in the variable `m_E`.

```
m_E = rho_M * 4 * pi / 3 * R_E^3
```

```
m_E =
```

$$\frac{4\,\pi\,{R_E}^3\,\rho_M}{3}$$

Next, we compute the density integral symbolically. We obtain

```
m = 4 * pi * int(r^2 * (rho_0 - b * r), r, 0, R_E)
```

```
m =
```

$$\frac{\pi\,{R_E}^3\,(4\,\rho_0 - 3\,R_E\,b)}{3}$$

Now we have defined a system of two equations for the two unknowns. We solve for b and ρ_0 by using the `solve` function.

```
sol = solve([m == m_E, rho_0 - b * R_E == rho_E], [b, rho_0])
```

```
sol = struct with fields:
        b: -(4*(rho_E - rho_M))/R_E
    rho_0: 4*rho_M - 3*rho_E
```

Next, we substitute the symbolic variables by their numerical values.

```
sol = subs(sol, [rho_E, rho_M, R_E], ...
    vpa([2700.0, 5515.0, 6.371e6]))
```

```
sol = struct with fields:
        b: 0.0017673834562862972845707110343745
    rho_0: 13960.0
```

Gravitational attraction inside the Earth

It is known that the gravitational potential inside a sphere with a radially symmetric density function is

$$V(r) = -\frac{4\pi f}{r}\int_0^r \rho(r')r'^2\,dr' - 4\pi f\int_r^{R_E}\rho(r')r'\,dr'.$$

The first integral amounts for the potential at the surface of a sphere with radius r, whereas the second computes the potential taken at the inner surface of a hollow sphere.

```
syms f r_p % better: r_prime
i1 = -4 * pi * f / r * ...
    int((rho_0 - b * r_p) * r_p^2, r_p, 0, r)
```

```
i1 =
```

$$-\frac{\pi\, f\, r^2\, (4\,\rho_0 - 3\,b\,r)}{3}$$

```
i2 = -4 * pi * f * int((rho_0 - b * r_p) * r_p, r_p, r, R_E)
```

i2 =

$$4\,\pi\, f\, \left(\frac{b\,{R_E}^3}{3} - \frac{\rho_0\,{R_E}^2}{2} - \frac{b\,r^3}{3} + \frac{\rho_0\,r^2}{2}\right)$$

We store the expression for the potential in a symbolic function `V(r)`.

```
V(r) = simplify(i1 + i2)
```

V(r) =

$$\frac{\pi\, f\, \left(4\,b\,{R_E}^3 - 6\,\rho_0\,{R_E}^2 - b\,r^3 + 2\,\rho_0\,r^2\right)}{3}$$

The gravitational attraction is

$\mathbf{g}(r) = -\nabla V(r) = -g(r)\hat{\mathbf{r}}$,

the scalar value $g(r)$ can be evaluated as symbolic function `g(r)`.

```
g(r) = -diff(V(r), r)
```

g(r) =

$$-\frac{\pi\, f\, \left(4\,r\,\rho_0 - 3\,b\,r^2\right)}{3}$$

For convenience, we define an anonymous function `gnum(r)`, with the help of which we can evaluate the gravitational attraction numerically.

```
gnum = @(r) subs(g(r), [rho_0, b, f, pi], ...
    [sol.rho_0, sol.b, vpa(6.674e-11), vpa(pi)]);
```

Where is the maximum of g(r)?

At this point we might ask, whether the gravitational attraction has an extremum inside the Earth. The answer is given by computing that depth for which the derivative of $g(r)$ with respect to r is zero.

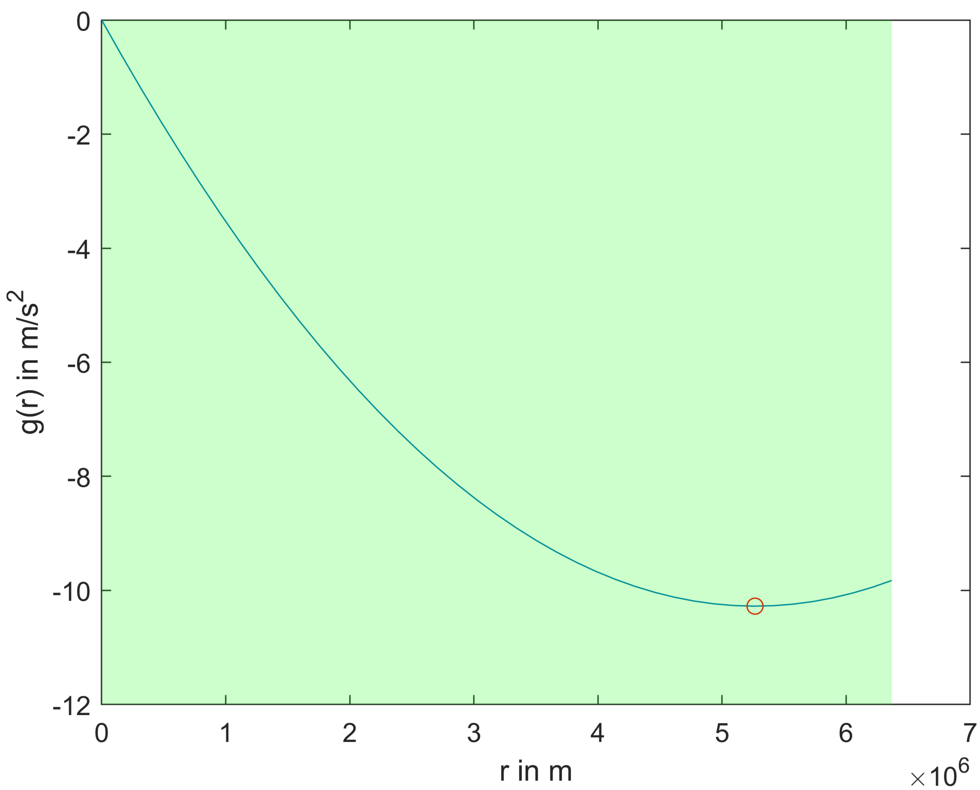
```
rextrem = subs(solve(diff(g(r), r) == 0, r), [rho_0, b], [sol.rho_0, sol.b])
```

rextrem = $5265788.0402605091770278271166371$

Visualization of the result

```
rr = linspace(0, 6.367e6, 41);

plot(rr, gnum(rr), ...
    rextrem, gnum(rextrem), 'ro');
xlabel('r in m')
ylabel('g(r) in m/s^2')
ylim([-12 0])
hold on
fill([0 0 6.367e6 6.367e6], [-12 0 0 -12], 'g', 'FaceAlpha', 0.2, 'EdgeColor','none')
hold off
```



Moment of inertia

The moment of inertia can be inferred from a thin spherical shell of radius a,

$$I = \frac{2}{3}ma^2.$$

The contribution of a small mass dm is therefore

$$dI = \frac{2}{3}a^2 dm$$

The total moment of inertia can be calculated with the integral

$$I = \frac{2}{3}\int_{0}^{R_E} \rho(r) r^2 dV = \frac{8\pi}{3}\int_{0}^{R_E} \rho(r) r^4 dr,$$

with the help of which we compute the moment of inertia factor α, a dimensionless number, where

$$\alpha = \frac{I}{mR_E^2}.$$

This factor is 0.4 for a solid sphere:

```
subs(...
    8 * pi / 3 * int(rho_M * r^4, r, 0, R_E) / (m_E * R_E^2), ...
    [R_E, rho_0, b, rho_M], ...
    [6.371e6, sol.rho_0, sol.b, 5515])
```

ans =

$$\frac{2}{5}$$

α shall be smaller for a sphere when there is an increased density with depth.

For our density model, we observe

$$\alpha \approx 0.3319,$$

which is slightly more than reported in the literature, where $\alpha = 0.3307$ (Williams, 1994).

```
subs(...
    8 * pi / 3 * int((rho_0 - b * r) * r^4, r, 0, R_E) / (m_E * R_E^2), ...
    [R_E, rho_0, b, rho_M], ...
    [6.371e6, sol.rho_0, sol.b, 5515])
```

ans = $0.33194318525234209731036566938652$