Electrostatic Dipole

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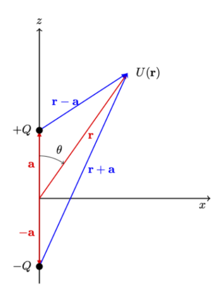
Problem description

```
imshow( ...
    imread('https://user-images.githubusercontent.com/6057466/103615215-7845e780-4f2a-
    'InitialMagnification','fit', 'Interpolation',"bilinear");
```



Two charges are fixed at the z-axis. The positive charge $+Q$ is located at $\mathbf{a} = (0, 0, a)^\top$, while the negative charge $-Q$ is located at $-\mathbf{a} = (0, 0, -a)^\top$. The point of observation is $\mathbf{r}$, $r = |\mathbf{r}|$. We ask for the potential $U = U(\mathbf{r})$.

Textbook approach

Exploiting the principle of superposition we obtain for the potential

$$U(\mathbf{r}) = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{|\mathbf{r}-\mathbf{a}|} - \frac{1}{|\mathbf{r}+\mathbf{a}|}\right].$$

For $|\mathbf{a}| \ll |\mathbf{r}|$ the inverse distance can be expanded in a series

$$\frac{1}{|\mathbf{r}\pm\mathbf{a}|} \approx \frac{1}{r}\left\{1 \pm 2\frac{\mathbf{a}\cdot\mathbf{r}}{r^2}\right\}^{-1/2} \approx \frac{1}{r}\left[1 \mp \frac{\mathbf{a}\cdot\mathbf{r}}{r^2}\right].$$

Thus,

$$\left[\frac{1}{|\mathbf{r}-\mathbf{a}|} - \frac{1}{|\mathbf{r}+\mathbf{a}|}\right] = \frac{1}{r}\left[1 + \frac{\mathbf{a}\cdot\mathbf{r}}{r^2} - \left(1 - \frac{\mathbf{a}\cdot\mathbf{r}}{r^2}\right)\right] = \frac{2\mathbf{a}\cdot\mathbf{r}}{r^3}.$$

Let's denote $\mathbf{p} = 2Q\mathbf{a}$ the *electric dipole moment*. The dipole moment $\mathbf{p}$ points *from the negative to the positive charge*!

We observe that U, because of the scalar product, depends on both r and θ:

$$U(r,\theta) = \frac{Q}{4\pi\epsilon}\frac{2\mathbf{a}\cdot\mathbf{r}}{r^3} = \frac{p}{4\pi\varepsilon}\frac{\cos\theta}{r^2}.$$

Recall that

$$\nabla\frac{1}{r} = -\frac{\mathbf{r}}{r^3}.$$

The dipole potential can be obtained by taking the derivative of the monopol potential with respect to the coordinates of the monopol potential in the direction of the dipole axis. Note that the term $\mathbf{p}\cdot\nabla$ denotes a *directional derivative*.

We obtain

$$U(\mathbf{r}) = -\frac{1}{4\pi\varepsilon}\mathbf{p}\cdot\nabla\frac{1}{r}.$$

Legendre polynomials approach

Following the notion of the *generating function* of the Legendre polynomials, we get

$$U(r) = \frac{Q}{4\pi\varepsilon r}\left\{\left[1 - 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2\right]^{-1/2} - \left[1 + 2\frac{a}{r}\cos\theta + \left(\frac{a}{r}\right)^2\right]^{-1/2}\right\}$$

and, after Taylor expansion,

$$U(r) = \frac{Q}{4\pi\varepsilon r}\left[\sum_{n=0}^{\infty}\left(\frac{a}{r}\right)^n P_n(\cos\theta) - \sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{r}\right)^n P_n(\cos\theta)\right].$$

The terms with even powers cancel each other because $(-1)^{2n} = 1$.

Thus, we obtain

$$U(r) = \frac{2Q}{4\pi\varepsilon r}\left[\left(\frac{a}{r}\right)P_1(\cos\theta) + \left(\frac{a}{r}\right)^3 P_3(\cos\theta) + \ldots\right].$$

For $a \ll r$ the term with $P_1(\cos\theta)$ dominates, and we obtain the potential of the electrostatic dipole

$$U(r) = \frac{p}{4\pi\varepsilon}\frac{P_1(\cos\theta)}{r^2}.$$

Recall that the Legendre polynomial of degree $n = 1$ is

$$P_1(\cos\theta) = \cos\theta.$$

This reproduces the general textbook equation, which we derived above:

$$U(r,\theta) = \frac{p}{4\pi\varepsilon}\frac{\cos\theta}{r^2}.$$

Electric field

The electric field can be obtained from the potential by applying the gradient, i.e.,

$$\mathbf{E} = -\nabla U.$$

```
syms U r theta
U(r, theta) = cos(theta) / r^2
```

```
U(r, theta) =
```

$$\frac{\cos(\theta)}{r^2}$$

```
gradient_sph = ...
    [diff(U, r), ...
    1 / r * diff(U, theta)];

E = -gradient_sph
```

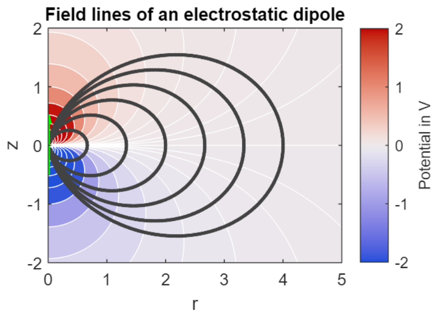
```
E(r, theta) =
```

$$\left(\frac{2\cos(\theta)}{r^3} \quad \frac{\sin(\theta)}{r^3}\right)$$

Visualization of the potential and its gradient

```
x = linspace(-5, 5, 200);
z = linspace(-2, 2, 80);
[X, Z] = meshgrid(x, z);
pot = Z ./ sqrt(X.^2 + Z.^2).^3;
dx = uniquetol(diff(x));
[Ex, Ez] = gradient(-pot, dx);
clines = logspace(-2, 2, 15);
cl = [sort(-clines) 0 clines];
contourf(x, z, pot, cl, 'Linewidth', 0.1, 'Color', 'w');
caxis([-2, 2])
cb = colorbar();
ylabel(cb, 'Potential in V')
[XX, ZZ] = meshgrid(linspace(0, 4, 7), zeros(1, 7));
l = streamline(x, z, Ex, Ez, XX, ZZ);
set(l, 'Color', '#424242', 'Linewidth', 2)
l = streamline(x, z, -Ex, -Ez, XX, -ZZ);
set(l, 'Color', '#424242', 'Linewidth', 2)
hold on
q = quiver(0, -0.5, 0, 1, 1);
q.Color = 'g';
q.LineWidth = 2;
q.MaxHeadSize = 0.4;
```

```
hold off
colorcet('D1');
t = title('Field lines of an electrostatic dipole');
t.FontSize = 10;
axis image
xlim([0, 5])
xlabel('r')
ylabel('z')
```



The electric field varies with r and θ, i.e., it is directional. It is

$$\mathbf{E}(r,\theta) = \begin{pmatrix} E_r \\ E_\theta \end{pmatrix} = \frac{p}{4\pi\varepsilon r^3}\begin{pmatrix} 2\cos\theta \\ \sin\theta \end{pmatrix}.$$

Note that $\mathbf{E} = E_r \frac{\mathbf{r}}{r} + E_\theta \boldsymbol{\theta}$.

Verification of the results

As usual, we derive the main results using the *Symbolic Math Toolbox*.

First, we verify the statement

$$\left\{1 \pm 2\frac{\mathbf{a}\cdot\mathbf{r}}{r^2}\right\}^{-1/2} \approx 1 \mp \frac{\mathbf{a}\cdot\mathbf{r}}{r^2}$$

We set $b = \mathbf{a}\cdot\mathbf{r}$, expand in a Taylor series and neglect terms of higher order.

```
syms r b
taylor(1 / sqrt(1 + 2 * b / r^2), b, 'Order', 2)
```

ans =

$$1 - \frac{b}{r^2}$$

```
taylor(1 / sqrt(1 - 2 * b / r^2), b, 'Order', 2)
```

ans =

$$\frac{b}{r^2}+1$$

Next, we verify the result

$$\left[\frac{1}{|\mathbf{r}-\mathbf{a}|}-\frac{1}{|\mathbf{r}+\mathbf{a}|}\right] \approx \frac{1}{r}\frac{2\mathbf{a}\cdot\mathbf{r}}{r^2}$$

using b as introduced above:

```
1 / r * (taylor(1 / sqrt(1 - 2 * b/r^2 ), b, 'Order', 2) - ...
    taylor(1 / sqrt(1 + 2 * b / r^2 ), b, 'Order', 2))
```

```
ans =
```

$$\frac{2\,b}{r^3}$$