Laplace equation in spherical coordinates

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The Laplace equation in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}=0.$$

We first restrict the solution such that $u=u(r)$.

Euler's differential equation

The radial part of the Laplace equation leads to *Euler's differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}u}{\mathrm{d}r}\right)=2r\frac{\mathrm{d}u}{\mathrm{d}r}+r^2\frac{\mathrm{d}^2u}{\mathrm{d}r^2}=0.$$

We solve this equation with the *ansatz*

$$u(r)=r^\alpha,\quad u'(r)=\alpha r^{\alpha-1},\quad u''(r)=(\alpha-1)\alpha r^{\alpha-2}$$

Substitute this in the original equation, we obtain the *characteristic equation*

$$\begin{aligned}&2r\alpha r^{\alpha-1}+r^2(\alpha-1)\alpha r^{\alpha-2}=0\\&2\alpha u+(\alpha-1)\alpha u=0\\&\alpha(\alpha+1)u=0.\end{aligned}$$

The solution can be obtained using the *Symbolic Math Toolbox*.

We get

```
syms r alpha U(r)
solve(alpha * (alpha + 1) * U == 0, alpha)
```

ans =

$$\begin{pmatrix}-1\\0\end{pmatrix}$$

$$\alpha_1=-1,\quad\alpha_2=0.$$

The solution of Euler's equation is therefore

$u(r) = C_1 r^{-1} + C_2 r^0.$

The same result will be obtained by integrating directly:

```
Du = diff(U, r);
D2u = diff(U, r, 2);
eqn = 2 * r * Du + r^2 * D2u == 0
```

```
eqn(r) =
```

$$r^2 \frac{\partial^2}{\partial r^2} U(r) + 2\, r \frac{\partial}{\partial r} U(r) = 0$$

```
dsolve(eqn)
```

```
ans =
```

$$C_2 + \frac{C_1}{r}$$

Homogeneous functions

Definition: Any function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is called homogeneous if it satisfies

$$f(t\mathbf{x}) = t^n f(\mathbf{x})$$

for some fixed n.

In a spherical coordinate system with

$$\begin{aligned} x &= r \cos\phi \sin\theta \\ y &= r \sin\phi \sin\theta \\ z &= r \cos\theta \end{aligned}$$

we replace $u(r)$ by the homogeneous function $u_n(x, y, z)$

$$\begin{aligned} u_n(x, y, z) &= u_n(r \cos\phi \sin\theta, r \sin\phi \sin\theta, r \cos\theta) \\ &= r^n u_n(\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta) \\ &= r^n S_n(\phi, \theta) \end{aligned}$$

The $S_n(\phi, \theta)$ is referred to as a *spherical harmonic* of degree n.

If we now replace u by u_n in Euler's equation, we obtain

$$\begin{aligned} 2r \frac{\mathrm{d}u_n}{\mathrm{d}r} + r^2 \frac{\mathrm{d}^2 u_n}{\mathrm{d}r^2} &= 2rnr^{n-1} S_n + r^2 n(n-1) r^{n-2} S_n \\ &= 2nu_n + n(n-1)u_n \\ &= n(n+1)u_n. \end{aligned}$$

The function $u_n(r)$ and the factor $n(n+1)$ are the eigenfunctions and eigenvalues of the eigenvalue problem

$\Delta_r u_n = n(n+1)u_n.$

In the following we replace the radial part of the Laplace operator (i.e., all derivatives with respect to r) by $n(n+1)$.

Let's again turn to the full Laplace equation.

With $u_n = r^n S_n(\phi, \theta)$ we write

$$n(n+1)S_n + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\varphi^2} = 0.$$

$S_n(\phi, \theta)$ are spherical harmonics. They are the eigenfunctions of the angular part of the Laplacian in spherical coordinates.

We now consider problems with rotational symmetry with respect to a particular fixed axis.

The Laplace equation for S_n is then

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S_n}{\mathrm{d}\theta}\right) + n(n+1)S_n = 0.$$

We substitute $t = \cos\theta$ and obtain the Legendre differential equation for $P_n(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-t^2)\frac{\mathrm{d}P_n(t)}{\mathrm{d}t}\right] + n(n+1)P_n(t) = 0$$

where $S_n(\theta) = P_n(\cos\theta) = P_n(t)$.

The $P_n(t) = P_n(\cos\theta)$ are the Legendre polynomials. They are the solution of the Laplace equation in spherical coordinates for problems invariant to rotation (in our case about the fixed z-axis).

The associated Legendre differential equation

Now we turn to the solution of the complete Laplace equation.

So far we have considered zonal spherical harmonics of the form $S_n(\theta) = P_n(\cos\theta)$.

To obtain a solution of the form $S_n(\phi, \theta)$ we use the method of separation of variables, such that

$S_n(\theta, \phi) = \Theta(\theta)\Phi(\phi).$

We write

$$n(n+1)\Theta\Phi + \frac{\Phi}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \frac{\Theta}{\sin^2\theta}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = 0$$

or

$$n(n+1)\sin^2\theta + \frac{\sin\theta}{\Theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) = m^2$$

and

$$\frac{1}{\Phi}\frac{\mathrm{d}^2\Phi}{\mathrm{d}\phi^2} = -m^2.$$

The solution $\Theta(\theta)$ of the first equation (the general Legendre equation)

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

can be obtained by the substitution $t = \cos\theta$, $\Theta(\theta) = P_n^m(\cos\theta) = P_n^m(t)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-t^2)\frac{\mathrm{d}P_n^m(t)}{\mathrm{d}t}\right] + \left[n(n+1) - \frac{m^2}{1-t^2}\right]P_n^m(t) = 0$$

with $P_n^m(t) = P_n^m(\cos\theta)$ referred to as *associated Legendre polynomials*.

The second equation has the solution

$$\Phi(\phi) = A_m\cos m\phi + B_m\sin m\phi.$$

Associated Legendre polynomials

The associated Legendre polynomials $P_n^m(\cos\theta)$ can be computed by taking derivatives of the Legendre polynomials $P_n(\cos\theta)$.

It holds

$$P_n^m(t) = (1-t^2)^{\frac{m}{2}}\frac{\mathrm{d}^m}{\mathrm{d}t^m}P_n(t), \quad t = \cos\theta, \quad 0 \le m \le n$$

```
syms t n m theta
Pnm = @(t, n, m) ...
    (1 - t^2)^(m/2) * ...
    diff(legendreP(n, t), t, m);
```

Let's validate the result. For $n \le 2$ we obtain:

```
for i = 0:2
    for j = 0:i
        [i, j, subs(Pnm(t, i, j), t, cos(theta))]
    end
end
```

ans = $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$

ans = $\begin{pmatrix} 1 & 0 & \cos(\theta) \end{pmatrix}$

ans = $\begin{pmatrix} 1 & 1 & \sqrt{1-\cos(\theta)^2} \end{pmatrix}$

ans =

$\begin{pmatrix} 2 & 0 & \frac{3\cos(\theta)^2}{2} - \frac{1}{2} \end{pmatrix}$

ans = $\begin{pmatrix} 2 & 1 & 3\cos(\theta)\,\sqrt{1-\cos(\theta)^2} \end{pmatrix}$

ans = $\begin{pmatrix} 2 & 2 & 3 - 3\cos(\theta)^2 \end{pmatrix}$

Solution of the angular part

The solution of the angular part of the Laplace equation can be written as linear combination

$$S_n(\phi,\theta) = \sum_{m=0}^{n} \left(A_m^{(n)}\cos(m\phi) + B_m^{(n)}\sin(m\phi)\right) P_n^m(\cos\theta).$$

There are still $2n+1$ non-trivial unknown linear coefficients to be determined $(B_0 = 0)$.

Complete solution

If we multiply $S_n(\phi,\theta)$ with r^n we obtain a homogeneous polynomial of degree n in (x,y,z). The $2n+1$ terms comprising the linear combination are linear independent.

The Laplace spherical harmonics of degree n are composed of a linear combination of $2n+1$ spherical harmonics of degree n and order m with $m = 0, 1, \ldots, n$.

Spherical harmonics expansion

The Laplace spherical harmonics $S_n(\phi,\theta)$ form a complete set of orthonormal functions and thus form an orthonormal basis of the Hilbert space of square-integrable functions L^2. Any of these functions can be expanded as a linear combination of spherical harmonics

$$f(\phi,\theta) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} c_n^m Y_n^m(\phi,\theta)$$

where c_n^m are the spherical harmonic coefficients, Y_n^m is the corresponding spherical harmonic function.

Note: There are many conventions which are often a source of confusion.

For a comprehensive overview see this AGU Open Access article on "SHTools: Tools for working with Spherical harmonics".