# Legendre polynomials



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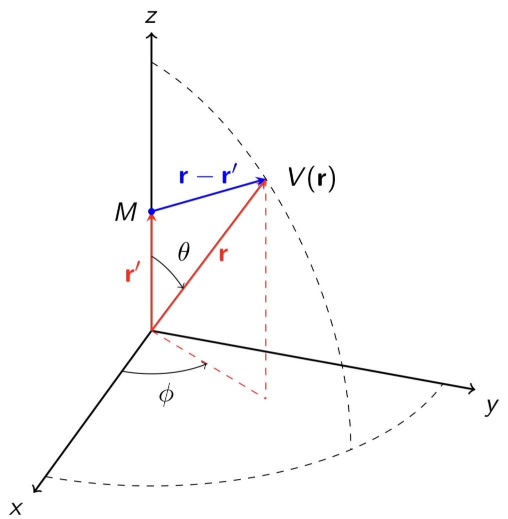
```
imshow( ...
    imread('https://user-images.githubusercontent.com/6057466/103557866-1a76b880-4eb4-11eb-9b5
    'InitialMagnification',24, 'Interpolation',"bilinear");
```




## Expansion of  the potential

In geophysics, we often have to deal with terms of the form $1/q = 1/|\mathbf{r} - \mathbf{r}'|$, e.g., in gravity or electrical resistivity methods.

If $|\mathbf{r}'| < |\mathbf{r}|$, we can expand the reciprocal distance $1/q$ in a Taylor series. To this end, we let $r = |\mathbf{r}|$ and $s = |\mathbf{r}'|$.

Then,

$$\frac{1}{q} = f(r,\theta) = \frac{1}{r}\left[1 + \left(\frac{s}{r}\right)^2 - 2\frac{s}{r}\cos\theta\right]^{-1/2}$$

with the help of the *law of cosines*.

With $b = \left(\frac{s}{r}\right)^2 - 2\frac{s}{r}\cos\theta$ we write

$$f(r,\theta) = \frac{1}{r}(1+b)^{-1/2}$$

Rearranging terms w.r.t. powers of $s/r$, we obtain

$$f(r,\theta) = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{s}{r}\right)^n P_n(\cos\theta)$$

where the $P_n(\cos\theta)$ are *Legendre polynomials* of degree $n$. The series is convergent for $r > s$.

If $r < s$, then we arrive at

$$f(r,\theta) = \frac{1}{s}\sum_{n=0}^{\infty}\left(\frac{r}{s}\right)^n P_n(\cos\theta), \qquad r < s.$$

The $P_n$ are independent of the azimuthal angle $\phi$ thus termed *zonal spherical harmonics*.

The function

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|}$$

is the *generating function* of the Legendre polynomials.

## Rodrigues' Formula

Here we demonstrate how the Legendre polynomials can be computed.

A very handy and especially compact expression for the Legendre polynomials is given by the *formula of Rodrigues*:

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.$$

It is straightforward to compute these polynomials with the gelp of MATLAB's *Symbolic Math Toolbox*.

We implement the above equation and substitute $x =: \cos\theta$.

```
syms x theta real
syms n positive
assume(n, 'integer')

Pn = @(x, n) subs( ...
    1 / (2^n * factorial(n)) * ...
    diff((x^2 - 1)^n, ...
    x, n), ...
    x, cos(theta));
for n = 0:3
    Pn(x, n)
```

```
end
```

ans = $1$

ans = $\cos(\theta)$

ans =

$$\frac{3\cos(\theta)^2}{2} - \frac{1}{2}$$

ans =

$$\cos(\theta)^3 + \frac{3\cos(\theta)\ (\cos(\theta)^2 - 1)}{2}$$

The $P_n$ are polynomials in $\cos\theta$, i.e., are constructed of powers of cosine, $\cos^n\theta$.

In the following, we reproduce the above result and first expand the inverse distance $f(r,\theta) = 1/q$:

```
syms s r theta f
```

```
f(r, theta) = 1 / r / ...
    sqrt(1 + (s/r)^2 - 2*s/r*cos(theta))
```

f(r, theta) =

$$\frac{1}{r\sqrt{\frac{s^2}{r^2} - \frac{2s\cos(\theta)}{r} + 1}}$$

With $b$ as introduced in the previous section, we obtain a polynomial in $b$, such that

$$f(r,b) = \frac{1}{r}\sum_{n=0}^{\infty}\alpha_n b^n.$$

```
syms b n
taylor(1 / r / sqrt(1 + b), b, 'Order', 5)
```

ans =

$$\frac{1}{r} - \frac{b}{2r} + \frac{3b^2}{8r} - \frac{5b^3}{16r} + \frac{35b^4}{128r}$$

The coefficients $\alpha_n$ are given by the *binomial coefficient*

$$\alpha_n = \binom{-1/2}{n}.$$

```
subs(nchoosek(-1/2, n), n, 0:4)
```

```
ans =
```

$$\left(1 \quad -\frac{1}{2} \quad \frac{3}{8} \quad -\frac{5}{16} \quad \frac{35}{128}\right)$$

A valid expression for the expanded potential up to order 3 is given by

```
collect(simplify(taylor(f, s, 'Order', 7), 'Steps', 100), 1/r)
```

```
ans(r, theta) =
```

$$\frac{1}{r}+\frac{s^6\ (231\cos(\theta)^6-315\cos(\theta)^4+105\cos(\theta)^2-5)}{r^7\ 16}+\frac{s\cos(\theta)}{r^2}+\frac{s^4\ (35\cos(\theta)^4-30\cos(\theta)^2+3)}{r^5\ 8}+\frac{s^2\ (}{}$$

$$f(r,\theta)=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{s}{r}\right)^n P_n(\cos\theta)$$

## Visualization of the Legendre polynomials on a unit sphere

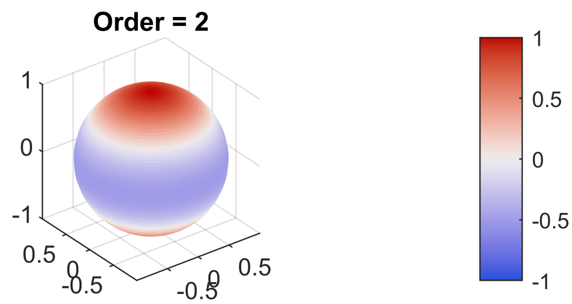
We use MATLAB to visualize the Legendre polynomials on a sphere. We use MATLAB's implementation `legendreP`.

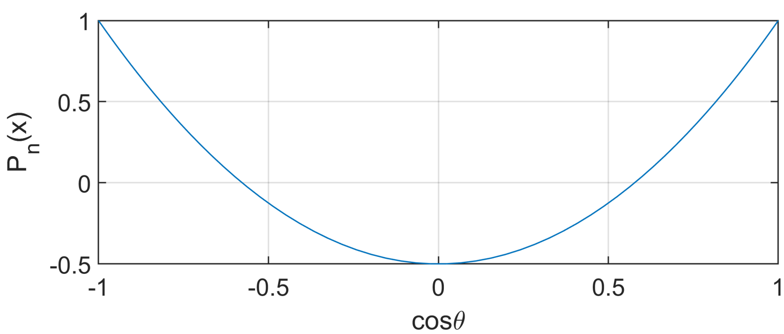
We make the following observations:

- Legendre polynomials $P_n$ are independent of the azimuthal direction
- the number of roots (zeros parallel to the equator) depends on the order $n$
- $P_n$ are an odd function of $\theta$ if $n$ odd, or an even function if $n$ even.

```
Phi = linspace(-pi, pi, 36);
Theta = linspace(0, pi, 72);
[PHI, THETA] = meshgrid(Phi, Theta);
PN = @(t, n) legendreP(n, cos(t));
order = 2;
DATA = PN(THETA, order);
DATA2 = sqrt(4*cos(THETA).^2 + sin(THETA).^2);
subplot(2, 1, 1)
[x,y,z] = sph2cart(PHI, THETA - pi/2, 1);
surf(x,y,z,DATA);
title(sprintf("Order = %d", order))
colorcet('D1');
caxis([-1 1])
axis equal
shading flat
colorbar();
subplot(2, 1, 2)
y = legendreP(order, cos(Theta));
h = plot(cos(Theta), y);
```

```
xlabel('cos\theta')
ylabel('P_n(x)')
grid();
box on
```




## More properties

The Legendre polynomials form an orthonormal system on $x \in [-1, 1]$, such that

$$\int_{-1}^{1} P_m(x)P_n(x)\mathrm{d}x = \frac{2}{2n+1}\delta_{mn}.$$

As an example, for $m = n = 3$ we obtain $2/7$.

```
syms y
Pnx = @(x, n) 1 / (2^n * factorial(n)) * diff((x^2 - 1)^n, x, n);

I = int(Pnx(y, 3) * Pnx(y, 3), y, -1, 1)
```

```
I =
```

$$\frac{2}{7}$$