Potential and gravitational attraction of a sphere

We consider a sphere of radius a centered at point $(0, 0, t)$, $t > 0$. The density contrast is $\Delta\rho$.

The excess mass caused by the sphere is therefore

$$m = \frac{4\pi}{3} a^3 \Delta\rho,$$

its potential is

$$V(x, y, z) = -f \frac{m}{\sqrt{x^2 + y^2 + (z - t)^2}}.$$

We restrict our measurements to profiles in the plane $x = 0$.

```
syms y z t f m real
V(y) = -f * m / sqrt(y^2 + (z - t)^2)
```

V(y) =

$$-\frac{f\,m}{\sqrt{(t - z)^2 + y^2}}$$

The gravitational attraction is

$$V_z(y) := -\frac{\partial V}{\partial z}.$$

For $z = 0$ we obtain

```
Vz(y) = subs(-diff(V, z), z, 0)
```

Vz(y) =

$$\frac{f\,m\,t}{(t^2 + y^2)^{3/2}}$$

The horizontal component of the attraction in $z = 0$ is

```
Vy(y) = subs(-diff(V, y), z, 0)
```

Vy(y) =

$$-\frac{f\,m\,y}{(t^2 + y^2)^{3/2}}$$

and its second derivative

```
Vyy = subs(-diff(V, y, 2), z, 0)
```

```
Vyy(y) =
```

$$\frac{3\,f\,m\,y^2}{(t^2+y^2)^{5/2}} - \frac{f\,m}{(t^2+y^2)^{3/2}}$$

```
Vzyy(y) = subs(diff(Vz, y, 2), z, 0);
assume(t > 0)
assume(f > 0)
assume(m > 0)
solve(Vzyy == 0)
```

```
ans =
```

$$\begin{pmatrix} -\frac{t}{2} \\ \frac{t}{2} \end{pmatrix}$$

The deflection points of V_z are at $\pm t/2$.

The local extrema of V_y are located at

```
solve(Vyy == 0, y)
```

```
ans =
```

$$\begin{pmatrix} -\frac{\sqrt{2}\,t}{2} \\ \frac{\sqrt{2}\,t}{2} \end{pmatrix}$$

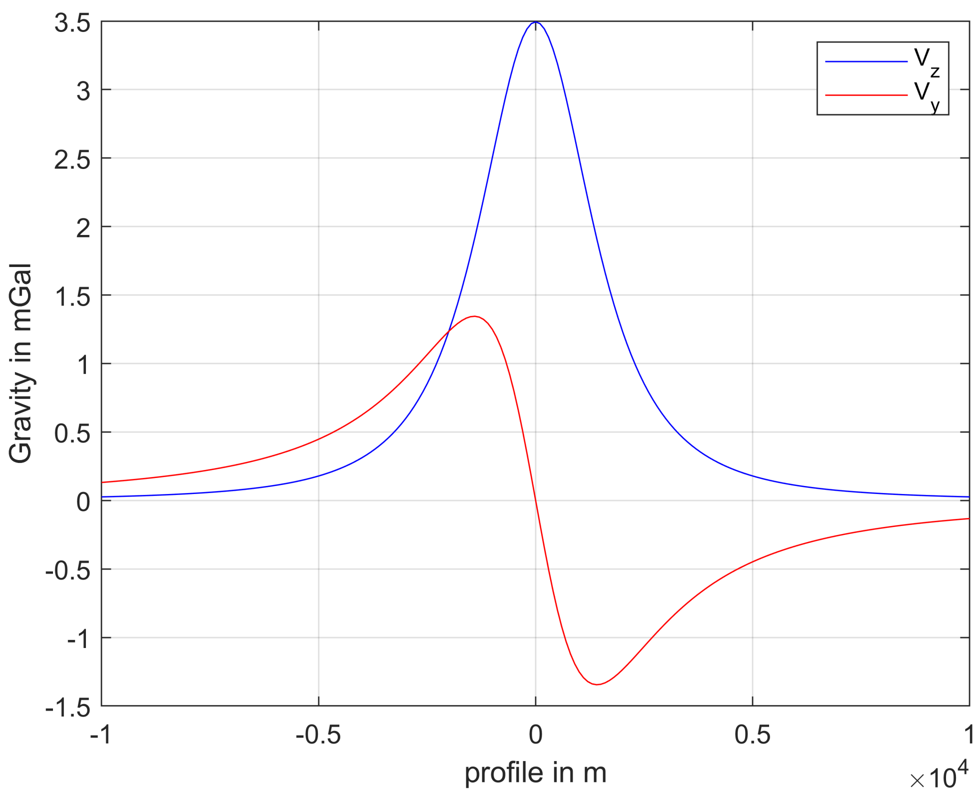
The extrema of V_y only depend on the depth of the sphere. The horizontal distance between the extrema is $\sqrt{2}\,t$.

Example

We consider as example a sphere at depth $t = 2000$ m with radius $r = 1000$ m. The density contrast is $\Delta\rho = 500$ kg/m^3. The vertical and horizontal components of the gravitational attraction along a profile at $z = 0$ running from $-10^4 \leq y \leq 10^4$ m have to be computed in units of mGal (milliGal).

```
yy = -1e4:100:1e4;
plot(yy, ...
    subs(...
    Vz(yy), ...
    [m, f, t], ...
    [1e5 * 4 * pi / 3 * 1e3^3 * 5e2, 6.674e-11, 2e3]), ...
```

```
    'b', ...
    yy, ...
    subs(...
    Vy(yy), ...
    [m, f, t], ...
    [1e5 * 4 * pi / 3 * 1e3^3 * 5e2, 6.674e-11, 2e3]), ...
    'r')
% xline([-0.5 * 2e3 0.5 * 2e3], '-b', ['t', 't'])
% xl = xline([-sqrt(2)/2 * 2e3 sqrt(2)/2 * 2e3], '--r', {'-t', 't'});
% xl(1).LabelVerticalAlignment = 'middle';
% xl(2).LabelVerticalAlignment = 'middle';
xlabel('profile in m')
ylabel('Gravity in mGal')
legend('V_z', 'V_y')
grid on
```



The deflection of a pendulum from the vertical direction due to the mass of the sphere is

$$\alpha = \tan^{-1}\frac{V_y}{\gamma_0 + V_z},$$

where $\gamma_0 = 9.81 \ \ m/s^2$ is a good estimate of the normal gravitational attraction at the surface of the Earth.

This is the fundamental idea of the *Eötvös* torsion balance.

With the sphere properties introduced above we obtain the following figure:

```
Vy_num = subs(Vy(yy), ...
    [m, f, t], ...
    [1e5 * 4 * pi / 3 * 1e3^3 * 5e2, 6.674e-11, 2e3]);
Vz_num = subs(Vz(yy), ...
    [m, f, t], ...
    [1e5 * 4 * pi / 3 * 1e3^3 * 5e2, 6.674e-11, 2e3]);
plot(yy, 3600 * 180 / pi * atan(Vy_num ./ (9.81e5 + Vz_num)))
ylabel('deflection in arc sec')
xlabel('profile in m')
grid on
```