Lecture: Standard Model Theory

2019, 3+1 h/week, 14 weeks

Content:

- detailed introduction to SM (no loops)
  - theory of YM theories
  - theory of chiral fermions
  - QCD and EWSM Lagrangian and conseq.
  - tree-level processes in QED vs. in EW theory, parity violation
  - flavour sector, CKM matrix, CP violation
  - neutrino masses (?)
- quantization of YM theories
- phenomenology including higher-order corrections
  - running couplings
  - $gg \to H$ Higgs production, $H \to \gamma\gamma$
  - ρ-parameter
  - a_μ^{EW} (?)

Goals:

- firm understanding of all SM aspects
- important processes/observables/phenomena
- QFT background: YM, chiral fermions, anomalies

1. Introduction

S.M.: - fantastic theory: - describes vast range of phenomena (chemistry, ~~QM, QED~~ atoms, solids, nuclei, particles, strong, weak int., Higgs)

- math. consistent, intrinsically complete ("renormal.")
- based on few principles: relativistic, local QFT + gauge inv. + spontan. sym. breaking + renormalizability

$$\mathcal{L} \overset{\text{schematic}}{=} -\frac{1}{4} F^{A\mu\nu} F^{A}_{\mu\nu} + \bar{\psi}_i \, i\gamma^\mu D_\mu \psi_i + |D^\mu \Phi|^2$$

→ gauge interact. with g. bosons, fermions, Higgs

$$+ \, y_{ijk} \bar{\psi}_i \psi_j \phi_k$$

→ "Yukawa" f - Higgs int.

$$- \mu^2 |\Phi|^2 - \lambda |\Phi|^4$$

→ Higgs potential / self interact.

structure fixed by these principles

free parameters : gauge coupl., y's, μ^2, λ

- contains QED, nonrel. QM etc as limits
- describes confinement of quarks, gluons into hadrons
- ~~asympt particles~~ asymptotic freedom at high energies
- ~~[illegible]~~
- describes detailed properties of Z and W bosons (relations masses, couplings, mixings)
- describes flavour transitions (Kaon decays, b-decays)
- describes parity and CP violation

(→ many CPV and flavour violating processes described by only 4 param.!)

- contains "miracles", which are expt. confirmed!

2. Gauge invariance and Yang-Mills theories

- one of the most profound principles of physics?
- artifact of our inefficient ability to describe spin 1?

2.1 ~~Classical~~ Gauge invariance in ED and QED

2.1.1 Classical ED

E - statics:

$$\partial_t(\text{all}) = 0$$

$$\vec{\nabla} \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\vec{\nabla} \underbrace{\phi}_{\text{"scalar potential"}}$$

"gauge" invariance: $\phi(\vec{x}) \to \phi(\vec{x}) + \text{const.}$

does not change phys. observables

M - statics:

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \vec{\nabla} \times \underbrace{\vec{A}}_{\text{"vector potential"}}$$

gauge invariance: $\vec{A}(\vec{x}) \to \vec{A}(\vec{x}) + \vec{\nabla}\Lambda(\vec{x})$

does not change observ.

ED:

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\Rightarrow \quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}$$

relativistic notation: ~~gauge invariance~~

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ & 0 & -B_z & +B_y \\ & & 0 & -B_x \\ \text{antisym.} & & & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad (A^\mu) = \left(\frac{\phi}{c}, \vec{A}\right)$$

gauge invariance: $A^\mu(x) \to A^\mu(x) + \partial^\mu \Lambda(x)$

does not change observables

Lagrangian: $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} \cdot j^{\mu} A_{\mu}$

Euler-Lagrange w.r.t. AS yields

$$\partial_{\nu} F^{\nu\mu} = \mu_0 j^{\mu} \qquad \text{(Maxwell eq.)}$$

note: → Lagrangian requires A^{μ}

note: $\int d^4x\, \mathcal{L}$ is gauge inv. since $\partial_{\mu} j^{\mu} = 0$

(partial integration ⤳ $\int d^4x\, j^{\mu} \partial_{\mu} \Lambda = 0$)

note: Maxwell fixes only dynamics of $F^{\mu\nu}$; j^{μ} is left open

⤳ no closed theory

2.1.2 Quantum mechanics

free Schröd. eq.:

$$i\hbar\, \partial_t \psi(x) = \frac{\hbar^2}{2m} (-i\vec{\nabla})^2 \psi(x)$$

⤳ requires def. of "complex phase of ψ" at diff. points

⤳ idea: allow for arbitrary phase changes

$$\psi'(x) = e^{-iq\theta(x)} \psi(x)$$

without changing physics

⤳ introduce "covariant derivative"

$$D'_{\mu} \psi'(x) = e^{-iq\theta(x)} D_{\mu} \psi(x)$$

with

$$D_{\mu} = \partial_{\mu} + iq A_{\mu}(x)$$

$$A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu} \theta(x)$$

↝ Schröd. eq.: $i\hbar(\partial_t + iqA_0)\psi = \frac{\hbar^2}{2m}(-i\partial_i + qA_i)^2\psi$

contains interaction!

↝ interaction is unphysical if one can always choose a gauge where $A_\mu(x) \equiv 0$

↝ there is an obstruction if $F_{\mu\nu} \neq 0$

⇒ this provides an alternative way to motivate ED and the existence of A^μ, $F^{\mu\nu}$.

2.1.3 QED

free Dirac eq. $(i\not{\partial} - m)\psi = 0$

$\not{\partial} = \partial_\mu \gamma^\mu$ — γ^μ: 4x4 matrices; ψ: 4-component spinor

free Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$

$$\left[\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0 \rightsquigarrow \text{Dirac eq.}\right]$$

(here: ψ is a dynamical field, not QM wave function) repeat argument, require local phase invariance:

introduce

$$D^\mu = \partial^\mu + iqA^\mu$$

write

$$\mathcal{L}_\psi = \bar\psi\,(i\underbrace{\not D}_{\gamma_\mu D^\mu} - m)\,\psi$$

then, under

$$\psi \to \psi' = e^{-iq\theta}\,\psi$$

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu\theta$$

we get $D'^\mu\psi' = (\partial^\mu + iqA^\mu + iq\partial^\mu\theta)\,e^{-iq\theta}\psi$
$= e^{-iq\theta}(\partial^\mu + iqA^\mu)\psi = e^{-iq\theta}D^\mu\psi$

$$\mathcal{L}_\psi \to \bar\psi'\,(i\not D' - m)\,\psi'$$
$$= \bar\psi\,e^{+iq\theta}\,(i\,e^{-iq\theta}\not D\psi - e^{-iq\theta}\psi)$$
$$= \mathcal{L}_\psi$$

add dynamics for A^μ: $\mathcal{L}_{gauge} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

$$\mathcal{L} = \bar\psi\,(i\not D - m)\,\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

QED!

taken as a class. field theory:

- describes dynamics of A^μ (Maxwell eq.)
  and of ψ (Dirac eq.) ~~with~~.
- interaction in Dirac eq. via D^μ
  in M. eq. via $j^\mu = -c\dfrac{\partial\mathcal{L}_\psi}{\partial A_\mu} = +cq\bar\psi\gamma^\mu\psi$
- character of a complete theory

quantization: field quanta of A^μ: photons, spin 1
transverse polarization

2.2 Lie groups and representations

General group: $G = \{g\}$ with

$$g_1 \circ g_2 \in G$$

$$\forall g \in G \ \exists g^{-1} : g^{-1} \circ g = g \circ g^{-1} = e$$

$$\exists e \in G : e \circ g = g \cdot e = g \quad \forall g \in G$$

- translations, rotations $SO(3)$, Lorentz/Poincaré, $U(1)$, $SU(N)$, ...
  crystal groups

representations: important in QM since linear structure

mapping $g \longmapsto R(g)$

abstract group element → g; linear operator/matrix → $R(g)$

such that $g_1 \circ g_2 = g_3 \longmapsto R(g_1) R(g_2) = R(g_3)$

same multipl. rule!

often: consider matrix groups, then the group itself is equal to a representation of itself

but there can be further representations!

e.g. representation of rotations on q.m. states $|j, m\rangle$ or on spinors.

Lie groups :

group elements = differentiable functions of real parameters θ^a $(a=1,\dots,n)$

(group = diff. manifold)

rotations def. by 3 Euler angles

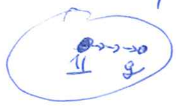
spatial translations def. by 3 variables $\Delta x, \Delta y, \Delta z$

Lie groups around identity:

consider small region around identity

function $g(\theta)$ with $\mathbb{1} = g(0)$

as ~~any element in region~~



$\cancel{g = g(\theta}$

Taylor : $g(\theta) = \underbrace{\mathbb{1} - i}_{\text{convention}}\ \theta^a \underbrace{t^a}_{\text{expansion coefficients}} + \mathcal{O}(\theta^2)$

written as expon. $= e^{-i\theta^a t^a}$

written as successive small transf. $= \lim\limits_{k\to\infty} \left(\mathbb{1} - i \frac{\theta^a}{k} t^a\right)^k$

commutation relation :

$g(\theta)\, g(\phi)\, g^{-1}(\theta)\, g^{-1}(\phi) = \mathbb{1} - (\theta^a t^a)(\phi^b t^b) + (\phi^b t^b)(\theta^a t^a) + \dots$

product = group element

$=: g(\xi)$ ← must be possible

$= \mathbb{1} - i\xi^a t^a + \mathcal{O}(\xi^2)$

Clearly : ξ = function of θ, ϕ, differentiable; for $\theta = 0$: $\xi = 0$, for $\phi = 0$: $\xi = 0$

$\Rightarrow \xi^c = C^c{}_{ab}\, \theta^a \phi^b$ + higher orders

compare $\theta^a\phi^b$-terms $\Longrightarrow$ $[t^a, t^b] = i\, C^c{}_{ab}\, t^c$

Lie algebra associated with G:

space spanned by t^a with

$$\boxed{[t^a, t^b] = i\, C^c_{ab}\, t^c \quad , \quad C^c_{ab} = - C^c_{ba}}$$

structure constants – define mult. law
– equivalent to knowing group structure

often: compact, simple Lie groups

$SU(N)$, $SO(N)$, $Sp(N)$, $E_6, E_8, \ldots$

- no subgroup commutes with the rest

semi-simple Lie groups

$SU(2) \times SU(2)$, $SU(3) \times SU(2)$ etc

non-semi-simple:

$SU(3) \times SU(2) \times U(1)$

abelian factor

non-compact:

Lorentz, Poincaré groups

for compact, semisimple Lie groups:

can assume normalization

$$\mathrm{Tr}(t^a t^b) = \lambda\, \delta^{ab} \quad , \lambda = \text{constant}$$

then: $C^c{}_{ab}$ is totally antisymmetric

write

$$\boxed{[t^a, t^b] = i\, f_{abc}\, t^c \quad , f_{abc} = \text{tot. antisym}}$$

representations of Lie group/Lie algebra:

$$g(\theta) \longmapsto R(\theta) = \mathbb{1} - i\,\theta^a T_R^a + O(\theta^2)$$

matrix or lin. operator

~~$g(\theta)g$~~

multipl. rule $\Rightarrow$ T_R^a also satisfy

$$\boxed{[T_R^a, T_R^b] = i f_{abc} T_R^c}$$

same comm. relation

any set of matrices ~~"repres. of Lie algebra~~

T_R^a with this relation constitutes a "repres. of the Lie Algebra"

[examples: fund., antifund., adjoint, $SU(2)$, $SU(3)$

see later]

2.3 Yang - Mills Theories

(classical)

2.3.1 Basic structure

Start: Assume "matter field" (scalar or spin 1/2)

$$\Phi = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_r \end{pmatrix}$$

, r components Φ_i (real or complex)

Assume transformation

$$\Phi(x) \longrightarrow \Phi'(x) = U(\theta(x))\, \Phi(x)$$

$U(\theta(x))$ = $r \times r$ - matrix

r - dimensional representation of some compact simple Lie group

$$= \mathbb{1} - i\, \theta^a(x)\, T^a + \mathcal{O}(\theta^2)$$

Lie algebra

$$[T^a, T^b] = i f_{abc}\, T^c$$

(if Φ = real : U should be orthogonal matrix,
- complex : " unitary)

require: covariant derivative D^μ: ansatz

$$D^\mu := \partial^\mu + ig\, A^\mu(x)$$

with $D^\mu \Phi(x) \longrightarrow U(\vartheta(x))\, D^\mu \Phi(x)$

solution:

$$ig\, A^\mu(x) \longrightarrow ig\, A'^\mu(x) = ig\, U(\vartheta(x))\, A^\mu(x)\, U^{-1}(\vartheta(x)) - (\partial^\mu U^{a}(\vartheta(x)))\, U^{-1}(\vartheta(x))$$

- inhomogeneous !
- second term $= i(\partial^\mu \vartheta^a(x))\, T^a$

$=$ lin.-comb. of T^a !

$\Rightarrow$ generally: $A^\mu = r \times r$ - matrix with decomposition

$$A^\mu(x) = A^{a\mu}(x)\, T^a$$

gauge invariant matter Lagrangian:

$$\mathcal{L}_{mat} = \begin{cases} |D^\mu \Phi|^2 & \text{scalar} \\ \bar{\Phi}\, i \not{D}\, \Phi & \text{spinor} \end{cases}$$

kinetic term for gauge field: "curvature"/field strength:

$$ig\, F^{\mu\nu} = [D^\mu, D^\nu] = ig(\partial^\mu A^\nu - \partial^\nu A^\mu) - g^2 [A^\mu, A^\nu]$$

$$= ig\, F^{a\mu\nu}\, T^a$$

- gauge covariant

$$F^{\mu\nu} \longrightarrow U(\vartheta)\, F^{\mu\nu}\, U^{-1}(\vartheta)$$

gauge invariant gauge kinetic Lag.:

$$\mathcal{L}_{gauge} = -\frac{1}{2} \mathrm{Tr}(F^{\mu\nu} F_{\mu\nu}) \quad (\text{if } \mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab} \text{ is used})$$

$$= -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu}$$

Full YM theory is defined by

$$\mathcal{L} = \mathcal{L}_{matter} + \mathcal{L}_{gauge}$$

can also study pure YM theory without matter

$$\mathcal{L} = \mathcal{L}_{gauge}$$

outlook:
- explicitly in components
- gauge transf. of $A^{a\mu}$ and $F^{a\mu\nu}$ actually indep. of choice of repres.
  $\Rightarrow$ even if many matter fields are used: all must have the same gauge coupling g
- SU(2), SU(3) examples