$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{-k^2+M^2} = \frac{i\pi^{d/2}}{(2\pi)^d} (M^2)^{d/2-1} \, \Gamma\left(-\frac{d}{2}+1\right)$$ $d = 4-2\varepsilon$

$$= i \underbrace{\frac{1}{16\pi^2}}_{\text{kombiniert mit}} \underbrace{(4\pi)^\varepsilon} \boxed{(M^2)^{d/2-1}} \boxed{\Gamma(-1+\varepsilon)}$$

$1+\varepsilon \log(4\pi) + \dots$ endlich

universell $1\times$ pro loop

$\frac{g^2}{16\pi^2} = \boxed{\frac{\alpha}{4\pi}}$ $\alpha = \frac{g^2}{4\pi}$

$1\times$ pro Loop

universell

interessanter Teil: $(M^2)^{d/2-1}$ M einzige dim. Skala

dimensionale Analyse $\frac{1}{M^2} = (M^2)^{-1+d/2}$ ↑ Dimension des Integrationsmasses

$(M^2)^{d/2-1}$ klar aus Massendimensionsgründen

einzige nicht-triviale Faktor $\boxed{\Gamma(-1+\varepsilon)}$

$$\Gamma(-1+\varepsilon) = -\frac{1}{\varepsilon} - 1 + \gamma_E - \left(1 - \gamma_E + \frac{\gamma_E^2}{2}\right)\varepsilon + \dots$$

Auftreten von Euler-Mascheroni Konstante γ_E ist überflüssig, im Sinne trägt nicht in finalen endlichen Ergebnissen bei.!

Besser! $e^{-\gamma_E \varepsilon} \left(\underline{\Gamma(-1+\varepsilon) e^{\gamma_E \varepsilon}} \right)$

$$= e^{-\gamma_E \varepsilon} \left(-\frac{1}{\varepsilon} - 1 - \Big(1 + \underbrace{\frac{\pi^2}{12}}_{\frac{\zeta_2}{2}}\Big)\varepsilon + \dots \right)$$

$\boxed{\zeta_2 = \frac{\pi^2}{6}}$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{-k^2+M^2} = \underbrace{\frac{i}{16\pi^2}}_{i\frac{\alpha}{4\pi}} \underbrace{(4\pi)^\varepsilon e^{-\gamma_E \varepsilon}}_{N(\varepsilon)} \left\{ -\frac{1}{\varepsilon} - 1 - \left(1 + \frac{\pi^2}{12}\right)\varepsilon \right\} (M^2)^{-1+d/2}$$

$\lim_{\varepsilon \to 0} N(\varepsilon) = 1$

Literatur: Viele Integrale bekannt!

Angegeben: nicht-triviale Teil:
$\mathcal{N}\left(-\frac{1}{\varepsilon}\cancel{*}\, \cancel{d} - \left(1+\frac{\pi^2}{12}\right)\varepsilon\right)$

<u>Normierung</u>: $e^{-\gamma_E \varepsilon}$ oder $\Gamma(1+\varepsilon)$

$$\frac{e^{-\gamma_E\varepsilon}}{\Gamma(1+\varepsilon)} = 1+\underbrace{\frac{\pi^2}{12}}_{\frac{\zeta_2}{2}}\varepsilon^2+\dots \quad \Rightarrow \text{Unterschiede ab } \varepsilon^2$$

↳ Divergenzen der 4-dimensionalen Integrale entsprechen Polen in ε

$$\boxed{\int\frac{d^dk}{(2\pi)^d}\frac{1}{(-k^2+M^2)^\alpha} = \frac{i}{16\pi^2}\mathcal{N}\left(M^2\right)^{-\alpha+d/2}\frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)}}$$

Frage: Wie behandeln wir Produkte im Nenner?

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{A\cdot B}\,,\qquad \int\frac{d^dk}{(2\pi)^d}\frac{1}{ABC}$$

<u>Feynmanparameter</u>:
$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{\left[A(1-x)+Bx\right]^2} \qquad x \leftrightarrow (1-x)$$

$$\frac{1}{ABC} = \int_0^1 dx\int_0^x dy\,\frac{2!}{\left[A(1-x)+B(x-y)+Cy\right]^3}$$

$$\frac{1}{A_1^{\alpha_1}A_2^{\alpha_2}\cdots A_n^{\alpha_n}} = ?$$

Schwinger Param.: $\displaystyle\frac{1}{A^\alpha} = \int dt\,\frac{t^{\alpha-1}e^{-At}}{\Gamma(\alpha)}$ folgt aus Def. der Γ-Funktion

Schwinger-Parametrisierung

$$\prod_{i=1}^{n} A_i^{-\alpha_i} = \int_0^\infty dt_1 \cdots dt_n \; t_1^{\alpha_1-1} \cdots t_n^{\alpha_n-1} \, e^{-\sum A_i t_i} \frac{1}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)}$$

α-Parametrisierung (t)

$$\boxed{\int_0^\infty \frac{d\lambda}{\lambda}\,\delta\!\left(1-\frac{t}{\lambda}\right) = 1}$$

$$\prod_{i=1}^{n} A_i^{-\alpha_i} = \int_0^\infty dt_1 \cdots dt_n \frac{t_1^{\alpha_1-1} \cdots t_n^{\alpha_n-1} \, e^{-\sum A_i t_i}}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)} \underbrace{\int \frac{d\lambda}{\lambda}\,\delta\!\left(1-\frac{\sum t_i}{\lambda}\right)}_{=1}$$

$$\boxed{\text{def. } x_i := \frac{t_i}{\lambda}} \quad = \int \frac{d\lambda}{\lambda} \int dx_1 \cdots dx_n \frac{x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1}}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)} \lambda^{\sum \alpha_i} \delta\!\left(1-\sum x_i\right) e^{-\lambda \sum A_i x_i}$$

$\Downarrow$

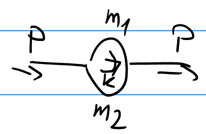
$$\boxed{\prod_{i=1}^{n} A_i^{-\alpha_i} = \frac{\Gamma(\sum \alpha_i)}{\Gamma(\alpha_1)\cdots\Gamma(\alpha_n)} \int_0^1 dx_1 \cdots dx_n \; x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} \frac{\delta(1-\sum x_i)}{\left[\sum A_i x_i\right]^{\sum \alpha_i}}}$$

Feynman Parametrisierung (Feynman Parameter x_i)

$$\frac{1}{A_1 A_2} = \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \int_0^1 dx_1\, dx_2 \frac{\delta(1-x_1-x_2)}{(A_1 x_1 + A_2 x_2)^2}$$

$$= \int_0^1 dx \frac{1}{\left[A_1(1-x) + A_2 x\right]^2}$$

konkretes Beispiel: Zweipunktfunktion



B_0	—o—
C_0	$\triangleleft$
D_0	$\square$

$$B_0(m_1^2, m_2^2, p^2) = \int_k \frac{1}{(k^2-m_1^2)\left((k+p)^2-m_2^2\right)}$$

F.P.

$$= \int_k \int_0^1 dx \frac{1}{\left[(k^2-m_1^2)(1-x) + \left((k+p)^2-m_2^2\right)x\right]^2}$$

$$= \int_k \int_0^1 dx \left[\underbrace{k^2 + 2k\cdot p\, x}_{\text{will Quadrat}} - m_1^2(1-x) + (p^2-m_2^2)x\right]^{-2}$$

$k \to k - px$

$$= \int_k \int_0^1 dx \left[\underbrace{k^2 - p^2 x - m_1^2(1-x) + (p^2 - m_2^2)x}_{-\Delta(x)} \right]^{-2}$$

$$= \int_0^1 dx \int_k \left[k^2 - \Delta(x) \right]^{-2}$$

μ^2 eingeführt um dimensionsloses in $()^\varepsilon$ zu bekommen

$$\Rightarrow B_0(m_1^2, m_2^2, p^2) = \frac{i(4\pi)^\varepsilon}{16\pi^2} \int_0^1 dx \left(\frac{\mu^2}{\Delta(x)} \right)^\varepsilon \Gamma(\varepsilon)$$

Entwickle in ε:

$$= \frac{i(4\pi)^\varepsilon}{16\pi^2} e^{-\gamma_E \varepsilon} \int_0^1 dx \left(\frac{1}{\varepsilon} + \log\left(\frac{\mu^2}{\Delta(x)} \right) \right)$$

$$= \frac{i(4\pi)^\varepsilon e^{-\gamma_E \varepsilon}}{16\pi^2} \left(\frac{1}{\varepsilon} + \int_0^1 dx \log\left(\frac{\mu^2}{\Delta(x)} \right) \right) + \mathcal{O}(\varepsilon)$$

Pol ist unabh. vom Impuls !

$$\frac{1}{k^2 - m^2 + i\varepsilon}$$

Müssen noch $\int \log$ ausführen

$$\Delta(x) = p^2 x^2 + m_1^2(1-x) - (p^2 - m_2^2)x$$

Spezialfall $m_1 = m_2 = m \Rightarrow \boxed{\Delta(x) = p^2(x^2 - x) + m^2 - i\varepsilon}$

1) $p^2 = 0 \rightsquigarrow$ o.B.d.A. $p = 0$

$$B_0(m, m, 0) = \dots \ln\frac{\mu^2}{m^2} = \frac{\partial}{\partial m^2} A_0(m^2)$$

2) $p^2 \ll m^2$: Entwickle in $\frac{p^2}{m^2}$

$$\log\frac{\Delta}{\mu^2} = \log\left(\frac{m^2}{\mu^2} \left(1 + \frac{p^2}{m^2}(x^2 - x) \right) \right)$$

$$\simeq \log\left(\frac{m^2}{\mu^2} \right) + \frac{p^2}{m^2}(x^2 - x) + \mathcal{O}\left(\frac{p^4}{m^4} \right)$$

$$B_0(m^2, m^2, p^2) \overset{p^2 \ll m^2}{=} -\log\frac{m^2}{\mu^2} - \int_0^1 \frac{p^2}{m^2}(x^2 - x) + \mathcal{O}\left(\frac{p^4}{m^4} \right)$$

$$= +\log\frac{\mu^2}{m^2} + \frac{1}{6}\frac{p^2}{m^2} + \mathcal{O}\left(\frac{p^4}{m^4} \right)$$

3) $|p^2| \gg m^2$ $\quad \log \frac{\Delta(x)}{\mu^2} = \log \frac{p^2(x^2-x) - i\varepsilon}{\mu^2}$

$p^2 < 0$: $\quad \log \frac{(-p^2)(x-x^2) - i\varepsilon}{\mu^2}$

$$= \log \frac{|p^2|}{\mu^2} + \log(x-x^2)$$

$p^2 > 0$: $\quad \log\left[\frac{p^2}{\mu^2}(x-x^2)(-1-i\varepsilon)\right]$

$$= \log \frac{p^2}{\mu^2} + \log(x-x^2) + \underbrace{\log(-1-i\varepsilon)}_{-i\pi}$$

Fasse beide Fälle zusammen

$$B_0(m^2, m^2, p^2) \overset{|p^2| \gg m^2}{=} \cdots - \log \frac{|p^2|}{\mu^2} + 2 + i\pi\,\Theta(p^2)$$

Auftreten des Imaginärteils erklärbar

Im (x — ○ — 1, A, B) $\hat{=}\ \Gamma(x \to A+B)$ $\quad$ daher $i\pi$ für $p^2 \to +\infty$

und nicht für $p^2 \to -\infty$

4) allgemeine Fall $m_1 = m_2 = m$

$$B_0(m^2, m^2, p^2) = \cdots + \int_0^1 dx \log\left(\frac{\mu^2}{p^2(x^2-x) + m^2}\right)$$

Faktorisieren des Polynoms im Nenner $\hat{=}$ Berechnet die Nullstellen

$$p^2(x^2-x) + m^2 = p^2(x-x_1)(x-x_2)$$

$$\boxed{y = \frac{m^2}{p^2} - i\varepsilon}$$

$p^2 > 0 \Rightarrow -\log\left(\frac{p^2(x^2-x) + m^2}{\mu^2}\right) = -\log\left(\frac{p^2}{\mu^2}\right) - \log\left[(x^2-x) + y\right]$

$$= -\log\frac{p^2}{\mu^2} - \log\left[(x-x_1)(x-x_2)\right] \qquad x_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1-4y}\right)$$

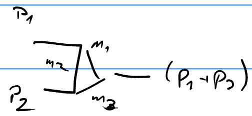
$$= \log\frac{\mu^2}{p^2} - \log(x-x_1) - \log(x-x_2)$$

$$\int_0^1 dx \left[\log\frac{\mu^2}{p^2} - \log(x-x_1) - \log(x-x_2)\right]$$

$$= \log\frac{\mu^2}{p^2} + \sum_j 1 - (1-x_j)\log(1-x_j) - x_j \log(-x_j)$$

$$= \log\frac{\mu^2}{p^2} + \sum_j 1 - (1-x_j)\log(1-x_j) - x_j \log x_j \qquad \mathrm{Im}\, x_1 = -\mathrm{Im}\, x_2$$

<u>Dreipunktfunktion C_0</u>



$p_1^2,\ p_2^2,\ (p_1+p_2)^2 = p_1^2+p_2^2+2p_1p_2$

oder $p_1^2,\ p_2^2,\ p_1 \cdot p_2$

$$C_0(p_1, p_2, m_1, m_2, m_3) =$$

$$= \int d^dk \frac{1}{(k^2-m_1^2)((k+p_1)^2-m_2^2)[(k+p_1+p_2)^2-m_3^2]}$$

Feynman Parameter
$$= \int_0^1 dx \int_0^1 dy \int d^dk \frac{2!}{\left[k^2 + 2k(p_1(x-y)+(p_1+p_2)y) + \text{Rest}\right]^3}$$

Shift $k \to k + p_1 x - p_1 y$

$$= \int_0^1 dx\,dy \int d^dk \frac{2!}{\left[k^2 - Q(x,y)\right]^3} \qquad Q(x,y) = (p_1 x + p_2 y)^2 + m_1^2(1-x) + m_2^2(x-y) + m_3^2 y + p_1^2(x-y) - (p_1+p_2)^2 y$$

$$\Rightarrow -\int_0^1 dx \int_0^1 dy \frac{\Gamma(1+\varepsilon)}{Q^{3-d/2}} \overset{\varepsilon\to 0}{=} -\int_0^1 dx \int_0^1 dy \frac{1}{Q(x,y)}$$

UV endlich aber Infrarot divergent (für gewisse m, p Kombinationen)

Spezialfall (Infrarote Divergenz)

$m_i = 0 \quad p_1^2, p_2^2 = 0 \qquad (p_1+p_2)^2 = p^2 \quad \Rightarrow \quad p_1 \cdot p_2 = \frac{1}{2} p^2$

Dann $Q(x,y) = p^2 xy - p^2 y = p^2 y (x-1)$

In d Dimensionen: $C_0 = - \int_0^1 dx \int_0^1 dy \, \frac{1}{Q} (4\pi)^\varepsilon \left(\frac{\mu^2}{Q}\right)^\varepsilon \Gamma(1+\varepsilon)$

Entwickle zuerst in ε und integriere dann!

$\Rightarrow -\frac{1}{p^2} \int_0^1 dx \int_0^1 dy \frac{1}{y(x-1)} = \infty \qquad \Rightarrow$ diese Strategie funktioniert nicht!

Also in d Dimensionen

$$\int_0^1 dx \int_0^{1*} dy \, Q^{-1-\varepsilon} = (-p^2)^{-1-\varepsilon} \int_0^1 dx \int_0^{1*} dy \, y^{-1-\varepsilon} (1-x)^{-1-\varepsilon}$$

$$= (-p^2)^{-1-\varepsilon} \int_0^1 dx \, (1-x)^{-1-\varepsilon} \left[\frac{y^{-\varepsilon}}{-\varepsilon}\right]_0^x$$

$$= -(-p^2)^{-1-\varepsilon} \frac{1}{\varepsilon} \underbrace{\int_0^1 dx \, (1-x)^{-1-\varepsilon} x^{-\varepsilon}}$$

Beta

Euler B-Funktion

$$= -(-p^2)^{-1-\varepsilon} \frac{1}{\varepsilon} B(-\varepsilon, 1-\varepsilon) \qquad B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dt \, t^{a-1} (1-t)^{b-1}$$

$$= -(-p^2)^{-1-\varepsilon} \frac{1}{\varepsilon} \frac{\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$= -(-p^2)^{-1-\varepsilon} \frac{1}{\varepsilon} \frac{(-\varepsilon)\Gamma(-\varepsilon)\Gamma(1-\varepsilon)}{(-\varepsilon)\Gamma(1-2\varepsilon)} = (-p^2)^{-1-\varepsilon} \frac{1}{\varepsilon^2} \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

von oben

$$C_0(0,0,p^2) = (4\pi\mu^2)^\varepsilon (-p^2)^{-1-\varepsilon} \boxed{\frac{1}{\varepsilon^2}} \frac{\boxed{\Gamma(1+\varepsilon)} \, \Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

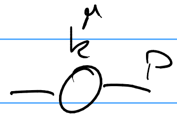
$(4\pi)^\varepsilon (-p^2)^{-1} \left(\frac{-p^2}{\mu^2}\right)^{-\varepsilon}$

Doppel IR Pol

Entw $\log\left(-\frac{p^2}{\mu^2}\right)$ Verzweigungsschnitt in $p^2 = 0$ bzw entlang pos Achse

Tensorintegrale

$$\int d^dk \, \frac{k^\mu}{(k^2-m^2)((k+p)^2-m^2)} \;\hat{=}$$



Passarino - Veltman Reduktion

einzig verfügbare 4-Vektor

$$\int d^dk \, \frac{k^\mu}{(k^2-m^2)((k+p)^2-m^2)} = p^\mu \, B_1(m,m,p^2)$$

Multipliziere beide Seiten mit p^μ

$$p^2 B_1(m,m,p^2) = \int d^dk \, \frac{p\cdot k}{(k^2-m^2)((k+p)^2-m^2)}$$

$$= \int d^dk \, \frac{\frac{1}{2}\left[(p+k)^2-m^2\right] - \frac{1}{2}\left[k^2-m^2\right] - \frac{1}{2}p^2}{(k^2-m^2)((k+p)^2-m^2)}$$

$$= \frac{1}{2}\int d^dk \, \frac{1}{(k^2-m^2)} - \frac{1}{2}\int d^dk \, \frac{1}{((k+p)^2-m^2)} - \frac{1}{2}\int d^dk \, \frac{p^2}{(k^2-m^2)((k+p)^2-m^2)}$$

$$= \frac{1}{2} A_0(m^2) - \frac{1}{2} A_0(m^2) - \frac{1}{2} p^2 B_0(m,m,p^2)$$

$$B_1(m,m,p^2) = -\frac{1}{2} B_0(m,m,p^2)$$

einschleifen

$\Rightarrow$ alle Tensorintegrale lassen auf skalare Integrale reduzieren

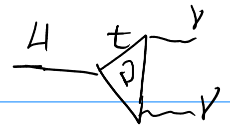
höherer Tensorrang

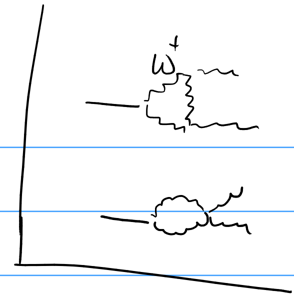
$$\int \frac{k^\mu k^\nu}{(k^2-m^2)((k+p)^2-m^2)} = p^\mu p^\nu \, B_{22}(p^2,m,m) + g^{\mu\nu} \, B_{21}(p^2,m,m)$$

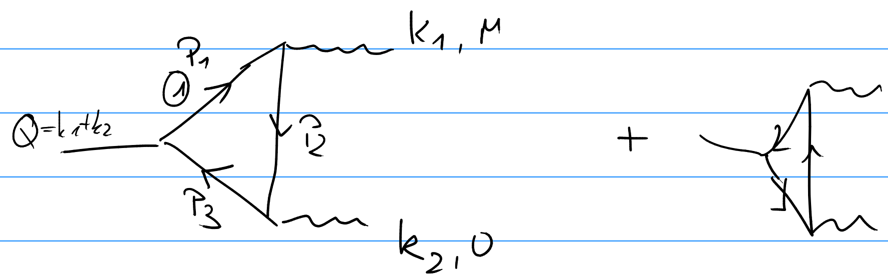
Dreipunktfunktion Rang 3

$$\int \frac{k^\mu k^\nu k^\rho}{P_1 \; P_2} = p_1^\mu p_1^\nu p_1^\rho \, C_{111} + p_1^\mu p_1^\nu p_2^\rho \, C_{112} + \dots \; g^{\mu\nu} p_1^\rho \, C_{001} \; \dots$$

$H \to \gamma\gamma$







$$A^{\mu\nu} = \int d^d l \; Tr\left\{ (\not{p}_1 + m)(\not{p}_3 + m)\gamma^\nu (\not{p}_2 + m)\gamma^\mu \right\} \Big/ (p_1^2 - m^2)(p_3^2 - m^2)(p_2^2 - m^2)$$

Was ist die Struktur dieser Amplitude?

$$A^{\mu\nu} = k_1^\nu k_2^\mu A + g^{\mu\nu} B + k_1^\mu k_2^\nu C + k_1^\mu k_1^\nu D + k_2^\mu k_2^\nu \underset{E(0,0,m,M_H)}{E(k_1^2, k_2^2, m, M_H)}$$

QED Ward Identitäten: $k_{1\mu} A^{\mu\nu} = k_{2\nu} A^{\mu\nu} = 0$

$$k_{1\mu} A^{\mu\nu} = k_1 k_2 k_1^\nu A + k_1^\nu B + \underbrace{k_1^2}_{=0} k_2^\nu C + \underbrace{k_1^2}_{=0} k_1^\nu D + k_1 k_2 k_2^\nu E = 0 \quad (1)$$

$$k_{2\nu} A^{\mu\nu} = k_2^\mu k_1 k_2 A + k_2^\mu B + k_1^\mu \underbrace{k_2 \cdot k_2}_{=0} C + k_1 k_2 k_1^\mu D + k_2^\mu \underbrace{k_2^2}_{=0} E = 0 \quad (2)$$

$$k_{1\mu} k_{2\nu} A^{\mu\nu} = (k_1 k_2)^2 A + k_1 k_2 B = 0 \quad (3)$$

$(3) \Rightarrow B = -k_1 k_2 A \quad (4)$

(4) in (1) und (2) $\Rightarrow E = 0, \; D = 0$

C beliebig, allerdings $\varepsilon^*_\mu \varepsilon^*_\nu A^{\mu\nu}$

daher trägt C zum Ergebnis nicht bei

Berechne $A \to B$ $\quad D, E = 0$ Check!

$$A^{\mu\nu} = (k_1^\nu k_2^\mu - k_1 k_2 g^{\mu\nu}) A$$

"Projektoren"

$$P_A^{\mu\nu} = \frac{(d-1)\, k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - k_1 k_2\, g^{\mu\nu}}{(d-2)(k_1 k_2)^2}$$

$$P_A^{\mu\nu} A^{\mu\nu} = A \qquad \text{entsprechend} \quad P_B^{\mu\nu} \ \dots \ P_E^{\mu\nu}$$

Vorteil: keine Tensorintegrale!

$$\Gamma(H \to \gamma\gamma) = \frac{M_H^3}{64\pi} |A_t|^2$$

$$A_t = \tilde{A} \frac{3}{2\tau} \left(1 + \left(1 - \frac{1}{\tau}\right) \arcsin^2 \sqrt{\tau}\right)$$

$$\tilde{A} = N_c \frac{2\alpha \sqrt{2} G_F}{3\pi} Q_f^2 \qquad \tau = \frac{M_H^2}{4m_t^2}$$

$$\Gamma(H \to \gamma\gamma) = \frac{M_H^3}{64\pi} |A_t + A_W|^2 \qquad \tau_W = \frac{M_H^2}{4M_W^2}$$

$$A_W = -\frac{\alpha\sqrt{2} G_F}{2\pi} \left(2 + \frac{3}{\tau_W} + \frac{3}{\tau_W}\left(2 - \frac{1}{\tau_W}\right) \arcsin^2 \sqrt{\tau_W}\right)$$

W Beitrag Faktor 4 grösser und negative Interferenz