Standard Model Theory – Tutorial 1

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1. **Work through gauge and BRS transformations in Yang-Mills Theories:**

a) The gauge transformation for a multiplet of fermion fields $\psi(x)$ is given by

$$\psi(x) \to U(x)\psi(x), \tag{1}$$

with a unitary matrix $U(x)$, the covariant derivative is given by

$$D^\mu \psi(x) = \left(\partial^\mu + igW^\mu(x)\right)\psi(x) \tag{2}$$

with a gauge-field matrix $W^\mu(x)$. The required gauge transformation of the covariant derivative is

$$D^\mu\psi(x) \to U(x)D^\mu\psi(x). \tag{3}$$

Hint: The infinitesimal versions of these equations using transformation parameters $\theta^a(x)$ and expanded in generators T^a are

$$\psi(x) \to \left(1 - i\theta^a(x)T^a\right)\psi(x), \tag{4}$$
$$W^\mu(x) = W^{a\mu}(x)T^a\,. \tag{5}$$

In all exercises, the generators satisfy the commutation relations

$$[T^a, T^b] = if^{abc}T^c \tag{6}$$

with structure constants f^{abc}.

Your task: Show that eq. (3) is satisfied (to first order in $\theta^a(x)$), if the gauge transformation of the gauge fields is given by

$$W^{a\mu}(x) \to W^{a\mu}(x) + \frac{1}{g}\partial^\mu\theta^a(x) - f^{abc}W^{b\mu}(x)\theta^c(x) \tag{7}$$

b) BRS transformations: In this exercise, $\phi(x)$ is a field multiplet which transforms according to a gauge group representation defined by generators T^a, and the gauge field is denoted by $A^\mu = A^\mu_a T^a$; ghost fields are introduced as $c = c_a T^a$, and similar for the antighost and B-field $\bar{c}$, B. The BRS transformations are defined as:

$$\begin{aligned}
sA^\mu(x) &= \partial^\mu c(x) - ig[c(x), A^\mu(x)]\ , && (8)\\
sA^\mu_a(x) &= \partial^\mu c_a(x) + gf_{abc}c_b(x)A^\mu_c(x)\ , && (9)\\
s\phi(x) &= -igc(x)\phi(x)\ , && (10)\\
s\phi_i(x) &= -igc_a(x)T^a_{ij}\phi_j(x)\ , && (11)\\
sc_a(x) &= \frac{1}{2}gf_{abc}c_b(x)c_c(x)\ , && (12)\\
sc(x) &= -igc(x)^2\ , && (13)\\
s\bar{c}_a(x) &= B_a(x)\ , && (14)\\
sB_a(x) &= 0\ . && (15)
\end{aligned}$$

s acts as a fermionic differential operator. Note that $s\phi$ is just an index-free way of writing $s\phi_i$; and note that sA^μ is equal to $sA^\mu_a T^a$.

Your tasks:

- Explain the relationship between these BRS transformations and the gauge transformations of the previous exercise.
- Point out that the fields can be grouped into three different kinds of fields, with three different kinds of BRS transformations.
- Show that s is nilpotent, $s^2 X = 0$ for all fields X. Do it for as many examples as you can, but at least for $s^2\phi$.
- Show that the choice of sc_a is equivalent to the nilpotency $s^2\phi = 0$ (provided all other BRS transformations are fixed).

2. **Work through tree-level processes:**

a) Write down the T-matrix element for the process $e^+e^- \to \mu^+\mu^-$ in QED at tree level. Use the Feynman rule for the photon propagator with arbitrary gauge parameter ξ. Write the result in the schematic form

$$\big(e^+e^-\text{-stuff}\big)^\mu \times \big(\mu^+\mu^-\text{-stuff}\big)^\nu \times \big(\text{photon propagator}\big)_{\mu\nu} \tag{16}$$

Your tasks:

- Verify the QED Ward identity $\big(e^+e^-\text{-stuff}\big)^\mu q_\mu = 0$, where q_μ is the photon momentum. You will need to apply some tricks and to use the Dirac equations of the spinors associated with the electron/positron.
- Use this to prove the ξ-independence of the T-matrix element.

b) Write down the T-matrix element for the QED scattering process $e^+e^- \to \gamma\gamma$ (there are two different Feynman diagrams, like for Compton scattering but "rotated by 90°"). Write the result in the schematic form

$$\epsilon^{\mu *}(k_1, \lambda_1)\epsilon^{\nu *}(k_2, \lambda_2)\mathcal{M}_{\mu\nu} \tag{17}$$

where k_i and λ_i are the momentum and polarization of the outgoing photon i.

Your tasks:

- Verify the QED Ward identity $\mathcal{M}_{\mu\nu}k_1^\mu = 0$. You will need to apply similar tricks as in the previous case.
- Use this to explain what happens when applying a Lorentz transformation $\Lambda^\mu{}_\nu$ to the T-matrix element: Clearly, at first all 4-momenta of all particles change by this Lorentz transformation, but the polarization vectors change in a different way, and one might worry that the T-matrix element is not Lorentz invariant. Explain that the behaviour of the transverse polarization vectors differs from a Lorentz transformation by a gauge transformation in the sense that for a transverse ϵ^μ (corresponding to massless momentum k and, say, $\lambda = 1$) we have that $\Lambda^\mu{}_\nu \epsilon^\nu = \epsilon'^\mu + a k'^\mu$ where ϵ' is a transverse polarization vector to momentum $k'^\mu = \Lambda^\mu{}_\nu k^\nu$ and a is a coefficient. Once you know this, you can use the Ward identity above to prove the Lorentz invariance of the T-matrix element.

c) b) Write down the T-matrix element for the QCD scattering process $q\bar{q} \to GG$ (this is essentially the same process as the previous QED process, except in QCD and replacing electrons by quarks and photons by gluons. In QCD there are three different Feynman diagrams, i.e. two QED-like diagrams and a new diagram which corresponds to the non-abelian nature of QCD). Write the result in the schematic form

$$\epsilon^{\mu *}(k_1, \lambda_1)\epsilon^{\nu *}(k_2, \lambda_2)\mathcal{M}_{\mu\nu} \tag{18}$$

where k_i and λ_i are the momentum and polarization of the outgoing gluon i.

Your tasks:

- Show that the QED-like Ward identity $\mathcal{M}_{\mu\nu}k_1^\mu = 0$ is **not satisfied**! The two QED-like diagrams do not satisfy it, and adding the third, non-abelian diagram does not help either. However, show that a slightly weaker Ward-like identity holds:

  $$\mathcal{M}_{\mu\nu}k_1^\mu \epsilon^{\nu *}(k_2, \lambda_2) = 0$$

  with a physical, transverse polarization vector $\epsilon^{\nu *}(k_2, \lambda_2)$.
- Explain that the validity of this Ward-like identity relies on the universality of the gauge coupling, i.e. on the fact that in all QCD vertices (gluon–quark and gluon–gluon) the same gauge coupling appears.

3. **Background information: Feynman rules** Using the BRS formalism one can consistently quantize Yang-Mills theories and derive Feynman rules for Green functions and S-matrix elements which lead to Lorentz invariant, causal and unitary results. The Feynman rules for the propagators are obtained from the free theory using either canonical or path integral quantization; the Feynman rules for interaction vertices can be read off from the Lagrangian as usual, and the Feynman rules for external lines can be obtained via LSZ reduction just like for scalar fields. The results can be found (partially) e.g. in Quigg, App. B and in Peskin/Schroeder, Fig. 16.1, 16.5, Eq. (16.29) (note different sign for gauge field and thus all vertices involving odd powers of gauge fields)

   The special rules for gauge field and ghost propagators are (for arbitrary ξ; but in most calculations you may use $\xi = 1$)

$$\text{gauge field propagator:} \qquad \frac{-i\left[g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}\right]}{q^2}\delta^{ab} \tag{19}$$

$$\text{ghost/anti-ghost propagator:} \qquad \frac{i}{q^2}\delta^{ab} \tag{20}$$

   The rules for wave functions for incoming/outgoing particles in S-matrix elements are

$$\text{incoming massless gauge boson:} \qquad \epsilon^\mu(k, \lambda) \tag{21}$$

$$\text{outgoing massless gauge boson:} \qquad \epsilon^{\mu *}(k, \lambda) \tag{22}$$

$$\text{incoming or outgoing ghost/anti-ghost:} \qquad 1 \tag{23}$$

where the $\epsilon^\mu(k,\lambda)$ are polarization vectors. For the case where the momentum $\vec{k}$ is oriented along the z-axis, we choose

$$\epsilon^\mu(k,1) = (0,1,0,0) \qquad \text{transverse, } x\text{-direction} \tag{24}$$

$$\epsilon^\mu(k,2) = (0,0,1,0) \qquad \text{transverse, } y\text{-direction} \tag{25}$$

$$\epsilon^\mu(k,3) = (0,0,0,1) \qquad \text{longitudinal, parallel to } \vec{k} \tag{26}$$

$$\epsilon^\mu(k,0) = (1,0,0,0) \qquad \text{timelike} \tag{27}$$

For general massless 4-momenta k with spatial momentum $\vec{k}$ we define the polarization vectors by a rotation: if $\vec{k}$ can be obtained by a 3-dimensional rotation R from some momentum in z-direction, then all four polarization vectors are defined by applying the same rotation R to their spatial components. We then have for any massless momentum k: $k^2 = 0$ and $k^\mu\epsilon_\mu(k,\lambda) = 0$ for $\lambda = 1,2$, but $k^\mu\epsilon_\mu(k,3) = -k^\mu\epsilon_\mu(k,0) \neq 0$. For a massless gauge boson, only the transverse polarizations ($\lambda = 1,2$) are physical; the longitudinal and timelike polarizations are unphysical.

In the following we specialize to QCD with gauge group SU(3) and quarks (which form SU(3) triplets) and provide a list of QCD vertices and colour algebra relations.

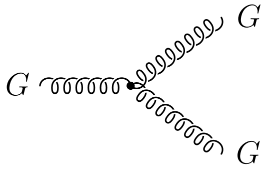
$$[T^a, T^b] = i f_{abc} T^c \ , \tag{28}$$

$$f_{abc} f_{dbc} = C(A)\delta_{ad} \ , \qquad C(A) = 3 \ , \tag{29}$$

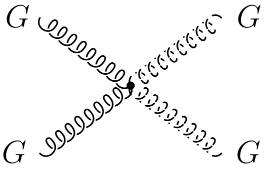
$$\mathrm{Tr}(T^a T^b) = T(F)\delta_{ab} \ , \qquad T(F) = \frac{1}{2} \ , \tag{30}$$

$$(T^a T^a)_{ij} = C(F)\delta_{ij} \ , \qquad C(F) = \frac{4}{3} \ . \tag{31}$$

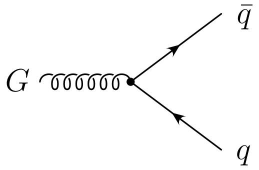
For the vertices, we use the notation $i\Gamma_{\varphi_i\varphi_j\ldots}(p_i, p_j, \ldots)$, which denotes the vertex with external fields $\varphi_i, \varphi_j, \ldots$ and with incoming momenta $p_i, p_j, \ldots$.



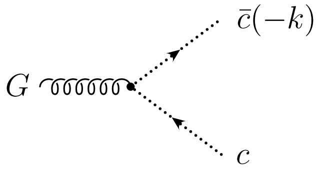
$$\begin{aligned} & i\Gamma_{G_a^\rho G_b^\mu G_c^\nu}(p_a, p_b, p_c) \\ = & -g f_{abc}[g_{\rho\mu}(p_a - p_b)_\nu \\ & \quad + g_{\mu\nu}(p_b - p_c)_\rho \\ & \quad + g_{\nu\rho}(p_c - p_a)_\mu] \end{aligned}$$



$$\begin{aligned} & i\Gamma_{G_a^\mu G_b^\nu G_e^\rho G_f^\sigma} \\ = & -ig^2 \\ & \Big[f_{abc}f_{efc}(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\nu\rho}) \\ & + f_{aec}f_{fbc}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\nu}g_{\rho\sigma}) \\ & + f_{afc}f_{bec}(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\sigma\nu})\Big] \end{aligned}$$



$$\begin{aligned} & i\Gamma_{G_a^\mu q_j \bar{q}_i} \\ = & -ig\gamma_\mu T^a_{ij} \end{aligned}$$



$$\begin{aligned} & i\Gamma_{c_c G_b^\mu \bar{c}_a}(k_1, k_2, -k) \\ = & \, igf_{bac}ik_\mu \end{aligned}$$