Gauge Theories and Standard Model – Tutorial 2

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1. **Unitarity of physical S-matrix:**

   Consider the process $\psi\bar{\psi} \to A_a^\mu(k_1)A_b^\nu(k_2)$ in a Yang-Mills theory with fermions, e.g. in QCD with one generic quark in the fundamental representation. The indicated colour indices and momenta correspond to the final state vector bosons. The corresponding T-matrix element has the structure

$$T_{fi}^{(\lambda_1\lambda_2)} = \mathcal{M}_{\mu\nu}\epsilon_1^{\mu*}(k_1,\lambda_1)\epsilon_2^{\nu*}(k_2,\lambda_2) \tag{1}$$

   where ϵ_i is the polarization vector of the gauge field i in the final state; these polarization vectors depend on the respective momentum and the indicated polarization states λ_i of the respective vector boson.

   Consider a further process $\psi\bar{\psi} \to c_a(k_1)\bar{c}_b(k_2)$ with the same initial state but ghost/antighost in the final state, and denote the T-matrix element as $\mathcal{M}$.

   a) Show that the following so-called Slavnov-Taylor identity holds at tree-level:

$$k_2^\nu\ \mathcal{M}_{\mu\nu} = \mathcal{M}\ k_{1\mu} \tag{2}$$

$$k_1^\mu\ \mathcal{M}_{\mu\nu} = \mathcal{M}\ k_{2\nu} \tag{3}$$

   (Note there are three Feynman diagrams for $\mathcal{M}_{\mu\nu}$ and one diagram for $\mathcal{M}$. It might be sufficient to do a partial calculation of these Feynman diagrams.)

   b) In the computation of probabilities, the sums over polarizations of squares of such amplitudes appear. You can assume the following polarization sums:

$$\text{phys. pol.:}\quad \sum_{\lambda=1,2}\epsilon^{\mu*}(k,\lambda)\epsilon^\rho(k,\lambda) = P^{\mu\rho}(k) \equiv -\left[g^{\mu\rho} - \frac{k^\mu\eta^\rho + \eta^\mu k^\rho}{k\cdot\eta} + \frac{k^\mu k^\rho}{(k\cdot\eta)^2}\right] \tag{4}$$

$$\text{all pol.:}\quad \sum_{\lambda=1,2,3,0}(-g_{\lambda\lambda})\epsilon^{\mu*}(k,\lambda)\epsilon^\rho(k,\lambda) = -g^{\mu\rho} \tag{5}$$

   Here η^μ is a constant vector, which can be chosen e.g. as $\eta = (1,0,0,0)$, and the symbol $(-g_{\lambda\lambda})$ (no summation here) originates from the negative norm of gluon states with timelike polarization. The result should not depend on this choice. Show:

$$P^{\mu\rho}(k_1)\ P^{\nu\sigma}(k_2)\ \mathcal{M}_{\mu\nu}\ \mathcal{M}^*_{\rho\sigma} = g^{\mu\rho}(k_1)\ g^{\nu\sigma}(k_2)\ \mathcal{M}_{\mu\nu}\ \mathcal{M}^*_{\rho\sigma} - 2\mathcal{M}\ \mathcal{M}^* \tag{6}$$

   I.e. in this sense, the unphysical polarizations of the vector fields cancel against the unphysical ghosts. This cancellation is required such that the S-matrix acting only on physical states is unitary. See e.g. Cheng/Li chapter 9.3, Aitchison/Hey chapter 13.5, Peskin/Schroeder chapter 16.3.

2. **Degrees of freedom:**

In classical electrodynamics we know that there are two linearly independent plane wave solutions to Maxwell's equations of the form $A^\mu(x) = \epsilon^\mu(k)e^{-ikx}$ with transverse polarization vectors $\epsilon^\mu(k)$.

How can we determine the physical degrees of freedom of a theory in the BRS formalism? In this problem, consider the full BRS Lagrangian, however in the limit $g = 0$ (i.e. the free non-interacting theory).

a) Consider plane wave solutions of the equations of motion $\propto e^{-ikx}$. Show: for a fixed gauge index a and fixed momentum k^μ, there are 6 (not 7 !) linearly independent solutions for the 7 fields A_a^μ, B_a, c_a, $\bar{c}_a$.

b) Show:

- two solutions are BRS invariant (i.e. the corresponding fields are annihilated by s in the limit $g = 0$) and *cannot* be written as BRS transforms of something else. These two solutions are by definition the physical degrees of freedom.
- there is a "BRS quartet" of two BRS non-invariant degrees of freedom and the two corresponding BRS transformations. All of these are by definition unphysical.

Note that this corresponds to the Kugo/Ojima-definition of $Ker(Q)/Im(Q)$ as the physical Hilbert space. For a further exposition of the full BRS+Kugo/Ojima formalism including higher orders see the original paper by Kugo/Ojima or the books by Kugo or Weinberg.

3. **Solution to previous exercise:**

a) $B \propto \partial A$, using e.o.m.. Hence, only 6 d.o.f. (i.e. we can impose independent initial conditions on 6 fields, but not on the 7th).

b) Use decomposition like in canonical quantization for A^μ, c, $\bar{c}$:

$$A^\mu(x) = \int d\tilde{k} \sum_{\lambda=0,1,2,3} \left(\epsilon^\mu(k,\lambda)a_\lambda(k)e^{-ikx} + h.c.\right) \tag{7}$$

$$c(x) = \int d\tilde{k} \left(c(k)e^{-ikx} + h.c.\right) \tag{8}$$

$$\bar{c}(x) = \int d\tilde{k} \left(\bar{c}(k)e^{-ikx} - h.c.\right) \text{ (note: antihermitian)} \tag{9}$$

Note: the BRS transformations may be viewed on the level of classical fields; or they may be viewed on the level of operators of the free, quantized theory. Then, $s\varphi = [Q_{BRS}, \varphi]_\pm$ where Q_{BRS} is the operator generating BRS transformations (it may be computed explicitly by deriving the conserved Noether charge to BRS invariance and replacing this by its quantum operator counterpart). In this sense we get a definition of BRS transformations for creation/annihilation operators and of states.

Result from $sA^\mu = \partial^\mu c$, $s\bar{c} = B = \partial^\mu A_\mu$ (using $\epsilon(k,0)k = -\epsilon(k,3)k = k^0 = |\vec{k}|$):

$$[Q_{BRS}, a_\lambda] = 0 \text{ (for } \lambda = 1,2) \tag{10}$$
$$[Q_{BRS}, a_0] = -ik^0 c \tag{11}$$
$$[Q_{BRS}, a_3] = -ik^0 c \tag{12}$$
$$[Q_{BRS}, \bar{c}]_+ = -ik^0(a_0 - a_3) \tag{13}$$
$$[Q_{BRS}, c]_+ = 0 \tag{14}$$

If we define $a_\pm = a_0 \pm a_3$, then we get

$$[Q_{BRS}, a_\lambda] = 0 \text{ (for } \lambda = 1,2) \tag{15}$$
$$[Q_{BRS}, a_+] = -2ik^0 c \qquad [Q_{BRS}, c]_+ = 0 \tag{16}$$
$$[Q_{BRS}, \bar{c}]_+ = -ik^0 a_- \qquad [Q_{BRS}, a_-] = 0 \tag{17}$$

For fun and clearer exposition we can also write it for states (using $Q_{BRS}|0\rangle = 0$)

$$Q_{BRS}|A(k,\lambda)\rangle = 0 \text{ (for } \lambda = 1,2) \tag{18}$$
$$Q_{BRS}|A(k,+)\rangle = -2ik^0|c(k)\rangle \qquad Q_{BRS}|c(k)\rangle = 0 \tag{19}$$
$$Q_{BRS}|\bar{c}(k)\rangle = -ik^0|A(k,-)\rangle \qquad Q_{BRS}|A(k,-)\rangle = 0 \tag{20}$$

This should drive home the point. There are two physical d.o.f., uniquely determined in the BRS+Kugo/Ojima formalism as the two transverse gluons; the other four d.o.f. form a BRS quartet, as it is called: two BRS doublets, which are connected by ghosts/antighosts.