Algorithmic Topology

3. X is compact $\Longleftrightarrow$ $(FIP) \Rightarrow \bigcap_{i\in I} C_i \neq \emptyset$

$[\Rightarrow]$ $\bigcap_{i\in I} C_i = X \setminus \bigcup_{i\in I}(X\setminus C_i)$ if this was empty, then

$X \subseteq \bigcup_{i\in I} \underbrace{(X\setminus C_i)}_{\text{open}} \Rightarrow \exists n\in\mathbb{N}, C_1,\dots,C_n \in \overset{(C_i)_{i\in I}}{\cancel{I}} : X \subseteq \bigcup_{i=1}^{n}(X\setminus C_i) \quad |(X\setminus)$

$\emptyset = X \setminus \bigcup_{i=1}^{n}(X\setminus C_i) = \bigcap_{i=1}^{n} C_i$ ↯

$[\Leftarrow]$ Let $(A_i)_{i\in I}$ be an open cover of X.

$X = \bigcup_{i\in I} A_i \quad |X\setminus()$

$\emptyset = X\setminus \bigcup_{i\in I} A_i = \bigcap_{i\in I}(X\setminus A_i)$. Contraposition of RHS yields that there is a finite subfamily with empty intersection:

$\exists n\in\mathbb{N}, A_1,\dots,A_n \in (A_i)_{i\in I}$

$\emptyset = \bigcap_{i=1}^{n}(X\setminus A_i) = X \setminus \bigcup_{i=1}^{n} A_i \Rightarrow \bigcup_{i=1}^{n} A_i = X$ ▨

5. 

$\left.\begin{matrix} V=0 \\ E=1 \\ F=1 \end{matrix}\right\} V-E+F = 0-1+1=0$

Def.: Given a set X, a topology on X is a set $\mathcal{O} \subseteq \mathcal{P}(X)$ that satisfies the following:

(i) $\emptyset, X \in \mathcal{O}$
(ii) $\forall I$ finite: $\bigcap_{i\in I} O_i \in \mathcal{O}$ whenever $\forall i\in I: O_i \in \mathcal{O}$
(iii) $\forall I$: $\bigcup_{i\in I} O_i \in \mathcal{O}$ whenever $\forall i\in I: O_i\in\mathcal{O}$

Elements of $\mathcal{O}$ are called open.

The good thing: You're not allowed to talk about distances.

Def.: A function $f: X\to Y$ is continuous if $\forall O \subseteq Y$ open, $f^{-1}(O) = \{x\in X : f(x)\in O\}$ is open.

- $X \xrightarrow{f} Y \xrightarrow{g} Z$

f continuous, g continuous
prove that $g\circ f$ is continuous.

Proof: Let $O \subseteq Z$ open.

$(g\circ f)^{-1}(O) = \underbrace{f^{-1}(\underbrace{g^{-1}(O)}_{\text{open as } g \text{ cont.}})}_{\text{open as } f \text{ cont.}}$ ▨

Def.: A homeomorphism ("isomorphism of topological spaces") is a bijective function $f: X\to Y$ s.t. f, f^{-1} are continuous.

- concatenations of homeomorphisms are homeomorphisms.
- $f: X\to Y$ bijective, continuous and closed $\Rightarrow$ f homeomorphism

Proof: Left to show that f^{-1} is continuous. Let A be open in X. Then A^c closed in X, so $f(A^c)$ is closed in Y, meaning $f(A^c)^c$ is open in Y:

$f(A^c)^c = \{y\in Y: y\notin f(A^c)\} \overset{f \text{ bij.}}{=} \{y\in Y: f^{-1}(y)\notin A^c\} = \{y\in Y: f^{-1}(y)\in A\} = \{y\in Y: y\in f(A)\}$
$= f(A)$

A topological space X is **path-connected** if $\forall x, y \in X$ $\exists f : [0,1] \to X$ continuous s.t. $f(0) = x$ and $f(1) = y$

A topological space is **disconnected** if it can be partitioned into (at least two) nonempty open sets.

- continuous images of connected topological spaces are connected.
- Are connected & path-connected equivalent?

Proof: Let X be a connected topological space. $f: X \to Y$ cont. Suppose we have $f(X)$ to be disconnected. Then there ex. a partition $(Y_i)_{i\in I}$ of $f(X)$ where Y_i are open and nonempty.

Since f is cont., $\forall i \in I$: $f^{-1}(Y_i)$ is open and nonempty. ($f: X \to f(X)$ is surj.)
They have pw. empty intersection, since if $x \in f^{-1}(Y_i)$ and $x \in f^{-1}(Y_j)$ with $i \neq j \in I$ ~~would~~ yield $f(x) \in Y_i, Y_j$ ↯ $(Y_i)_{i\in I}$ partition

~~So~~ Let x be in X. Then there exists a unique Y_i s.t. $f(x) \in Y_i$ $\Rightarrow x \in f^{-1}(Y_i)$. So $(f^{-1}(Y_i))_{i\in I}$ are a partition of nonempty open sets of X ↯ X connected. ▨

Let X be path-connected. Is it then connected?
Suppose we have partition $(X_i)_{i\in I}$ of X of nonempty open sets.
Let $i \neq j \in I$ and $x \in X_i$ and $y \in X_j$, f a continuous function $f: [0,1] \to X$ with $f(0) = x$ and $f(1) = y$.
Then $f^{-1}(X_i)$, $f^{-1}(X_j)$ are open in $[0,1]$. $f^{-1}(X_i)$ $f^{-1}(X_j)$ is nonempty since $f(0) = x \in X_i$ and $f(1) = y \in X_j$.
So $f^{-1}(X_i) = [0, r_1)$ and $f^{-1}(X_j) = (r_2, 1]$ for real $0 < r_1, r_2 < 1$.
$r_1 < r_2$ because $f^{-1}(X_i) \cap f^{-1}(X_j) = \emptyset$.

Chapter I: Point set topology

Def: Given a set X, a **topology on X** is a set $\tau \subseteq \mathcal{P}(X)$ s.t.

(i) $\emptyset, X \in \tau$
(ii) $\bigcap_{i\in I} O_i \in \tau$ $\forall I$ finite, $O_i \in \tau$ $\forall i \in I$
(iii) $\bigcup_{i\in I} O_i \in \tau$ $\forall I$ s.t. $O_i \in \tau$ $\forall i \in I$

Elements of τ are called **open**. Their complements are called **closed**.

A **basis** of a topology on X is a set $B \subseteq \mathcal{P}(X)$ satisfying (i) & (ii).

Fact: If B is a basis, then

$$\{O \subseteq X \mid O = \bigcup_{i\in I} O_i \text{ for some } i \in I \ \& \ O_i \in B\} \text{ is a topology on } X.$$

Example:

$$B = \{(a,b) \mid a < b,\ a,b \in [0,1]\} \cup \{[0,b), (a,1] \mid a,b \in [0,1]\} \cup \{[0,1]\}$$

Example: For any X, $\mathcal{P}(X)$ is a topology, the **discrete topology**.
$\{\emptyset, X\}$ is a topology, the **trivial topology**.

Def.: Given a set X with topology τ on X, $Y \subseteq X$. The subspace topology on Y is

$$\mathcal{S} = \{ Z \subseteq Y \mid Z = Y \cap O,\ O \in \tau \}$$

Prop.: The subspace topology is a topology on Y.

Proof: Check (i) - (iii).

(i) $\emptyset \in \tau \Rightarrow \emptyset = \emptyset \cap Y \Rightarrow \emptyset \in \mathcal{S}$.
$X \in \tau \Rightarrow Y = X \cap Y \Rightarrow Y \in \mathcal{S}$

(ii) Let $(U_i)_{i \in I} \subseteq \mathcal{S}$, I finite.
$U_i \in \mathcal{S}$, so $\exists\ O_i \in \tau$ s.t. $U_i = O_i \cap Y$
By (ii) for τ, $\bigcap_{i \in I} O_i \in \tau$.

$$\bigcap_{i \in I} U_i = \bigcap_{i \in I} (O_i \cap Y) = Y \cap \underbrace{\bigcap_{i \in I} O_i}_{\in \tau} \in \mathcal{S}$$

(iii) Let $(U_i)_{i \in I} \subseteq \mathcal{S}$, similar to (ii) ... ▨

Example: Let $X \subseteq \mathbb{R}^n$ is endowed with the subspace topology where the topology of $\mathbb{R}^n$ is given by the basis of Balls $B_r(x)$, $r \in \mathbb{R}_{>0}$, $x \in \mathbb{R}$

Recall: A function $f: X \to Y$ is continuous if $\forall\ O$ open in Y: $f^{-1}(O)$ is open.
If $Y \subseteq X$, consider the inclusion map

$$\iota : Y \to X$$
$$y \mapsto y$$

Fact: The subspace topology on Y is the crasest topology on Y so that ι is continuous.

Fact: Let D be endowed with the discrete topology. Then any map $f: D \to X$ with any topology on X is continuous.

Now let a topology on X be given and a surj. function $f: X \to Y$.
How should we define the "quotient topology" on Y?

$\Rightarrow$ Take the finest topology on Y s.t. f is continuous.

$\Rightarrow \{ U \subseteq Y \mid \exists\ O \subseteq X$ open with $f^{-1}(U) = O \}$

The disjoint union of two topological spaces (X, τ), $(Y, \mathcal{S})$ is defined as $(X \,\dot\cup\, Y, \{ O \mid O = O_1 \cup O_2,\ O_1 \in \tau,\ O_2 \in \mathcal{S} \})$

Prove that if X, Y are nonempty, then the disjoint union is not connected.
Assume $X \cap Y = \emptyset$.

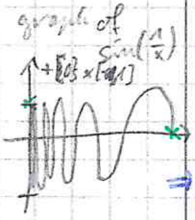
Proof: Let (X,τ), $(Y,\mathcal{S})$ be nonempty topological spaces. Def. $(X \cup Y, R)$ as the disjoint union.
Then $X \in R$ since $X = X \cup \emptyset$ where $X \in \tau$ and $\emptyset \in \mathcal{S}$ (def. top.)
and $Y \in R$ since $Y = \emptyset \cup Y$ where $\emptyset \in \tau$ and $Y \in \mathcal{S}$
and $X \cup Y$ then is a partition of nonempty open sets ▨

Prove that $[0,1]$ is connected.

Proof: We prove that it is path-connected instead, then the prop. follows.
Let $x, y \in X$. Def $\ell : [0,1] \to [0,1]$ by $\ell(s) = x + s(y-x)$. This is continuous becau

Claim: $X \cup Y$ is not path-connected.
Proof: Assume the opposite. Let $x \in X$ and $y \in Y$. Then there exists $f : [0,1] \to X \cup Y$ continuous s.t. $f(0) = x$, $f(1) = y$.

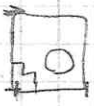
$X' := f^{-1}(X)$, $Y' := f^{-1}(Y)$ open and disjoint since X and Y are disjoint.
Since $X \cup Y$ covers $X \cup Y$, every $s \in [0,1]$ is in $X' \cup Y'$.
So $X' \cup Y'$ is a clopen partition of $[0,1]$ $\lightning$ ▨



Not every connected space is path-connected.

Prove that $[0,1]$ is connected.

⇒ product topology



use as basis →

Naive guess: X, Y top. spaces.
open sets of $X \times Y$ are products $O_1 \times O_2$; $O_1 \in \tau(X)$, $O_2 \in \tau(Y)$

Idea: $O \subseteq X \times Y$ is open iff $\pi_X(O) \in \tau(X)$, $\pi_Y(O) \in \tau(Y)$
use projection maps.

Def.:
Given X, Y top. spaces, the product topology of $X \times Y$ is given by the basis
$$\{ O_1 \times O_2 \mid O_1 \in \tau(X),\ O_2 \in \tau(Y) \}$$

Lemma: The product topology is indeed a topology.
Proof:
- $\emptyset$, $X \times Y \in$ prod. top. ✓
- closed under unions: by def. ✓
- finite intersections: suffices to check for pairwise intersections.

Let $O, U \in \tau(X \times Y)$ be given.
$$O = \bigcup_{i \in I} O_1^i \times O_2^i, \qquad U = \bigcup_{j \in J} U_1^j \times U_2^j$$
$$O \cap U = \Big(\bigcup_{i \in I} O_1^i \times O_2^i\Big) \cap \Big(\bigcup_{j \in J} U_1^j \times U_2^j\Big)$$
$$= \bigcup_{(i,j) \in I \times J} \big((O_1^i \times O_2^i) \cap (U_1^j \times U_2^j)\big)$$
$$= \bigcup_{(i,j) \in I \times J} \underbrace{(O_1^i \cap U_1^j) \times (O_2^i \times U_2^j)}_{\text{basic open set}}$$ ▨

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

π_X and π_Y continuous.
Easy: discrete topology on $X \times Y$.
Interesting: coarsest topology with that condition

For every $O \in \tau(X)$: $\pi_X^{-1}(O) = O \times Y$ open
$U \in \tau(Y)$: $\pi_Y^{-1}(U) = X \times U$ open
Unfortunately, this is not a topology:
- not closed under unions
- not closed under finite intersections

⇒ use as subbasis (close under unions and finite intersections)

space of infinite 0-1-sequences should have some topology:
0 1 0 0 1 1 0 1 0 1 ...

Two definitions:
① $O_1 \times O_2 \times O_3 \times \ldots$, O_i open in i-th copy of $\{0,1\}$ — discrete topology
$\{1\} \times \{1\} \times \{1\} \times \ldots$
product would not be compact!

② $X = \prod_{i \in I} X_i$

- $\pi_i : X \to X_i$ continuous $\Rightarrow X_1, X_2, \dots, O_i, X_{i+1}, \dots$
  $(x_1, x_2, \dots) \mapsto x_i$
- close under finite intersections and arbitrary unions

Def.: Given a family $(X_i \mid i \in I)$ of topological spaces, the **product topology** on $X = \prod_{i \in I} X_i$ is the coarsest topology s.th. all projection maps $\pi_i : X \to X_i$ are continuous, that is basic open sets are of the form:

$X_{i_1}, X_{i_2}, \dots, O_\alpha, \dots, X_{i_{k-1}}, O_k, \dots$

$\prod_{i \in I} Y_i$, $Y_i = X_i$ for all but finitely many $i \in I$ & $Y_i \in \mathcal{T}(X_i)$

Theorem Tychonov
Products of compact spaces are compact.

Why is this useful?
Compactness principle: (FIP)
$(A_i \mid i \in I)$ family of closed sets. If $\forall J \subseteq I$ finite : $\bigcap_{j \in J} A_j \neq \emptyset \Rightarrow \bigcap_{i \in I} A_i \neq \emptyset$

Intuition: build complicated object as element of $X = \prod_{i \in I} X_i$
have conditions
$y \in A_i \leftarrow$ closed | only need to check at finitely many points at once!

Fact Erdős, De Bruijn
Given $k \in \mathbb{N}$. If every finite subgraph of G is k-colourable, then G is k-colourable.

$\chi(G) =$ minimum of colors needed to color a graph s.t. adjacent vertices have different colors

$\chi(G) = \sup_{\substack{H \subseteq G \\ \text{finite}}} \chi(H)$

Proof: G a graph, every finite subgraph can be k-colored $\quad X = \prod_{v \in V(G)} \{1, \dots, k\} \quad k \in \mathbb{N}$
By Tychonoff, X is compact.
A_e colouring of G $\quad c : V(k) \to \{1, \dots, k\}$
s.t. colours at the endvertices of e are distinct
is closed. $\quad \{1, \dots, k\} \times \{1, \dots, k\} \setminus \{(a,a) \mid a \in \{1, \dots, k\}\}$

$(A_e \mid e \in E(G))$ has finite intersection property.
Let $e_1, \dots, e_s$ finitely many edges of G.
Let H be the subgraph of G with edge set $e_1, \dots, e_s$ with vertices $\bigcup_{1 \le r \le s} e_r$
H is k-colourable $\Rightarrow A_{e_1} \cap A_{e_2} \cap \dots \cap A_{e_s}$ is nonempty.
Compactness: $\bigcap_{e \in E(G)} A_e \neq \emptyset \Rightarrow \exists$ k-colouring of G ▨

Given a graph $G = (V, E)$, an **unfriendly partition** is a bipartition $V = A_1 \,\dot\cup\, A_2$ s.t. $\forall i \in \{0, 1\}$:
$\forall v \in A_i : \deg_{A_i}(v) \le \deg_{A_{1-i}}(v)$

Fact 1. Every finite graph admits an unfriendly partition.
- start with arbitrary partition
- if a vertex doesn't fit the criterion, move it to the other side
- iterate until all satisfy

count edges across, this number is monotonly increasing and bounded $\Rightarrow$ algorithm terminates

Conjecture Erdős 1960s

Every countable graph admits an unfriendly partition.

Fact 2: True for locally finite graphs by compactness.
$\uparrow$
$\forall v \in V: \deg v < \infty$

Proof: $X = \{\text{red}, \text{blue}\}^{V(G)} = \{ f : V(G) \to \{\text{red}, \text{blue}\}\}$
endow X with the product topology with $\{\text{red}, \text{blue}\}$ having discrete topology.
Tychonov $\Rightarrow$ X is compact.

Task: Define family $(A_i \mid i \in I)$ of closed sets s.t.
$f : V(G) \to \{\text{red}, \text{blue}\}$ is unfriendly iff $f \in \bigcap_{i \in I} A_i$.

For every $v \in V(G)$ define

$A_v := \{ f : V(G) \to \{\text{red}, \text{blue}\} \mid \underbrace{\deg_{f^{-1}(\text{colour of } v)} v \leq \deg_{f^{-1}(\text{not colour of } v)} v}_{}\}$ v is unhappy, i.e.

f is unfriendly iff $f \in \bigcap_{i \in I} A_i$

Lemma: A_v is closed.
Proof: $\deg v < \infty$ by assumption.
$O_v := \{ f : V(G) \to \{\text{red}, \text{blue}\} \mid v \text{ is happy}\}$ is a basic open set since v only has finitely many neighbours (only have to specify the finitely many colors of v and neighbours, rest is freely choosable).
$A_v = O_v^c$ is closed. ▨ We show for $W \subseteq V(G)$ finite:

$\bigcap_{v \in W} A_v \neq \emptyset$ since Fact 1 is applyable.
Indeed, let $H = G[W \cup \bigcup_{v \in W} N(v)]$ admits an unfriendly partition by Fact 1. Extend it arbitrarily to a partition of G. In this extension, all v's of W are unhappy.

By compactness: $\bigcap_{v \in V(G)} A_v \neq \emptyset$, yielding an unfriendly partition. ▨

- Is $\{0,1\}^V$ connected?
- X, Y top. spaces connected; is $X \times Y$ connected?
- " path-connected?
- Arbitrary products of connected spaces?

no, take for arbitrary $v \in V(G)$ the open partition $\{ f : V \to \{0,1\} \mid f(v) = 0\}$
$\cup \{ f : V \to \{0,1\} \mid f(v) = 1\}$

Let X, Y be connected. Assume $X \times Y$ is disconnected, so
$U_1 \cup U_2 = X \times Y$. ~~We find X_1, X_2, Y_1, Y_2 s.t. $U_1 = X_1 \times Y_1$ and $U_2 = X_2 \times Y_2$.~~
~~$X_1, X_2 \neq X$ because otherwise the other one would be empty. Same for Y_1, Y_2~~
Project onto X:
$\pi_X($

Let (X, τ) be a top. space.

Ⓐ (X, τ) is Fréchet if $\forall x \neq y \in X$ there exist neighbourhoods U_x of x and U_y of y with $x \notin U_y$ and $y \notin U_x$

Ⓑ (X, τ) is Hausdorff if $\forall x \neq y \in X$ there exist nbhds U_x of x and U_y of y with $U_x \cap U_y = \emptyset$.

Ⓒ (X, τ) is Kolmogorov if $\forall x \neq y \in X$ there is an open subset that either contains only x or only y.

$\overset{T_2}{\text{Hausdorff}} \overset{(1)}{\Rightarrow} \overset{T_1}{\text{Fréchet}} \overset{(2)}{\Rightarrow} \overset{T_0}{\text{Kolmogorov}}$

(1) Let (X,τ) be Hausdorff, $x \neq y \in X$. Use Hausdorff to obtain U_x, U_y open neighbourhoods with $U_x \cap U_y = \emptyset$.
Then $x \notin U_y$, because otherwise $x \in U_x \cap U_y = \emptyset$ $\lightning$
also $y \notin U_x$, ... $y \in U_x \cap U_y = \emptyset$ $\lightning$

(2) Let (X,τ) be Fréchet, $x \neq y \in X$. By Fréchet, obtain open nbhd's U_x, U_y with $x \notin U_y$ and $y \notin U_x$. Then use U_x which satisfies $x \in U_x \not\ni y$. ▨

2. a) Top. space that is not T_0: any space with trivial topology and at least two points

b) space that is T_0 but not T_1:
Choose $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, X\}$
Then (X,τ) is T_0 (always choose $\{a\}$).
Not T_1: Choose $x := a$, $y := b$. Then U_y has to be X, also containing x $\lightning$

space that is T_1 but not T_2: ~~$\{c\}$~~
~~Choose $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a,c\}, \{b,c\}, X\}$~~

~~$\tau = \{ \emptyset, x, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$~~

$\tau = \{ U \mid X \setminus U \text{ finite}\}$ with infinite X, e.g. $X = \mathbb{N}$
Not T_2: Let $x = 0$ and $y = 1$. Assume we have U_x, U_y open with $U_x \cap U_y = \emptyset$. We find finite C_x, C_y s.t. $U_x = \mathbb{N} \setminus C_x$ and $U_y = \mathbb{N} \setminus C_y$.
so: $U_x \cap U_y = (\mathbb{N} \setminus C_x) \cap (\mathbb{N} \setminus C_y) = (\mathbb{N} \cap C_x^c) \cap (\mathbb{N} \cap C_y^c)$
$= \mathbb{N} \cap (C_x^c \cap C_y^c) = \mathbb{N} \cap (C_x \cup C_y)^c = \mathbb{N} \setminus (C_x \cup C_y) = \emptyset \Rightarrow C_x \cup C_y = \mathbb{N}$ $\lightning$
C_x, C_y finite

But T_1: Let $x, y \in \mathbb{N}$. Choose $U_x := \mathbb{N} \setminus \{y\}$ and $U_y := \mathbb{N} \setminus \{x\}$.
Then U_x, U_y open, $x \in U_x$, $y \in U_y$, $x \notin U_y$ and $y \notin U_x$.

4. $T_1 \Leftrightarrow (\forall x \in X: \{x\} \text{ closed})$
[$\Rightarrow$] Let $x \in X$. For each $y_i \in X$ with $y_i \neq x$, we use Fréchet, obtaining U_{x_i} and U_{y_i} with $x \notin U_{y_i}$ and $y_i \notin U_{x_i}$.
Then $U := \bigcup_{\substack{y_i \in X \\ y_i \neq x}} U_{y_i}$ open with $x \notin U$ but $y \in U$ for every $y \neq x$, so $\{x\} = X \setminus U$ closed.

[$\Leftarrow$] Let $x, y \in X$, $x \neq y$. Choose $U_x := X \setminus \{y\}$ and $U_y := X \setminus \{x\}$ open (since $\{x\}, \{y\}$ closed), $x \in U_x$, $y \in U_y$, $y \notin U_x$, $x \notin U_y$ ▨

5. a) If (X,τ) Hausdorff and $(x_n)_{n \in \mathbb{N}}$ with $x_n \to x$ and $x_n \to y$ then $x = y$.
Convergence:
$\forall U$ open neighbourhood of x $\exists n_0 \in \mathbb{N}\, \forall n \geq n_0$: $x_n \in U$

Assume $x \neq y$. Use Hausdorff to obtain U_x, U_y open with $U_x \cap U_y = \emptyset$.
For U_x, we obtain by convergence a n_0 s.t. $\forall n \geq n_0$: $x_n \in U_x$
and U_y, we get a n_1 s.t. $\forall n \geq n_1$: $x_n \in U_y$.
Then follows for $m := \max(n_0, n_1)$: $x_m \in U_x \cap U_y$ $\lightning$ $= \emptyset$

b) Find Fréchet/Kolmogorov and series $(x_n)_{n \in \mathbb{N}}$ with $x_n \to x$, $x_n \to y$ and $x \neq y$.
Choose $X = \{a, b\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $x_n := a$ $(n \in \mathbb{N})$
Then $x_n \to a$ since x_n stays inside of $\{a\}$ resp. X
but also $x_n \to b$ since x_n stays in X, which is the only nbhd of b, but $a \neq b$.

Alternative: $X = \mathbb{N}$, cofinite topology, then $x_n := n$ $(n \in \mathbb{N})$ converges to every point

<u>Proof of Tychonov's Theorem</u> A top. space (X, τ) is **compact** if every cover has a finite subcover.

Equivalently, for every collection C of closed sets in X:
If the intersection of finitely many members of C is nonempty (FIP), then $\bigcap C \neq \emptyset$.

Tychonov: Let $\{X_i\}_{i \in I}$ be a fam. of top. spaces that are all compact.
Then $\prod_{i \in I} X_i$ compact.

Let S be a set and C a collection of subsets of S. Then we say C has **FIP** when every finite subcollection has nonempty intersection.

<u>Def.</u>: Let S be a set and $F \subseteq P(S)$. F is an **Ultrafilter** if

(1) F is closed under intersection, $\forall A, B \in F: A \cap B \in F$
(2) F is upwards closed, $\forall A \in F, B \subseteq S: A \subseteq B \Rightarrow B \in F$
(3) $\forall A \subseteq S$, either $A \in F$, $A^c \in F$, but not both.

<u>Note</u>: $\emptyset \notin F$.
<u>Observation</u>: F has FIP.

<u>Ultrafilter Lemma</u>: Let S be a set $C \in P(S)$ have the FIP.
Then $\exists$ Ultrafilter F on S s.t. $C \subseteq F$.

Let F be an Ultrafilter on a top. space X. We say that **$F \to x \in X$** if F contains all open neighbourhoods of x.

<u>Lemma O</u> Let U be a subset of a top. space X. Then U open in X iff every ultrafilter that converges to some $x \in U$ contains U.

<u>Proof</u>: $[\Rightarrow]$ Let $F \to x \in U$. Then U is an open nbhd of x, so $U \in F$.

$[\Leftarrow]$ Let $x \in U$. Suppose for a contradiction that U contains no open nbhd of x.
Define $C := \{\text{all open nbhds of } x\} \cup \{U^c\}$
C has FIP, use Ultrafilter lemma to get ultrafilter $F \supseteq C$.
Observe that $F \to x$, therefore it contains U by assumption, but also $U^c \in F$ ↯

$\rightsquigarrow \forall x \in U \; \exists$ open nbhd $U_x \ni x$ s.t. $U_x \subseteq U$
$U = \bigcup_{x \in U} U_x$ so U is open. ▨

<u>Lemma</u> A topological space X is compact iff every ultrafilter on X converges to at least one point in X.

<u>Proof</u>: $[\Rightarrow]$ Let F be an ultrafilter on X compact and suppose it doesn't converge. For each $x \in X$, let U_x be an open nbhd not contained in F. (contraposition of convergence)
The U_x cover X, so by compactness there exist $U_{x_1}, \dots, U_{x_n}$ with $X = \bigcup_{k=1}^{n} U_{x_k}$, $n \in \mathbb{N}$.
$\Rightarrow \bigcap_{k=1}^{n} U_{x_k}^c = \emptyset$ but since $U_{x_1}^c, \dots, U_{x_n}^c \in F$, also $\emptyset \in F$ ↯

$[\Leftarrow]$ For a contradiction, let $\mathcal{U}$ be an open cover of X without finite subcover.

$\mathcal{U}^c := \{U^c : U \in \mathcal{U}\}$ has FIP. By ultrafilter lemma, $\exists$ ultrafilter $F \supseteq \mathcal{U}^c$. By assumption, $\exists x \in X$ s.t. $F \to x$. $x \in U$ for some $U \in \mathcal{U}$. So $U \in F$ (convergence), but $U^c \in F$ by def. of F. ↯ ▨

Let X, Y be top. spaces and $f : X \to Y$. Let F be an ultrafilter on X. Then the pushforwards of F along f is $f_* F \in \mathcal{P}(Y)$ defined by $f_* F = \{S \subseteq Y : f^{-1}(S) \in F\}$

Lemma Let $f : X \to Y$. Then f continuous iff $\forall$ ultrafilters F on X and $x \in X$, $(F \to x \Rightarrow f_* F \to f(x))$

Proof [$\Rightarrow$] Let $x \in X$ with $F \to x$ an ultrafilter on X. Let U be an open nbhd of $f(x)$ in Y. Then $f^{-1}(U)$ is open in X and contains x. So $f^{-1}(U)$ an open nbhd of x, so $f^{-1}(U) \in F$, so $U \in f_* F$.

(otherwise, if $f^{-1}(U) = \emptyset$, we're done)

[$\Leftarrow$] Let $U \subseteq Y$ be open. Let $x \in X$ s.t. $f(x) \in U$. Let F be an ultrafilter on X s.t. $F \to x$. So $f_* F \to f(x)$. So $U \in f_* F$ (because U is open nbhd of $f(x)$). So $f^{-1}(U) \in F$ by definition.
Conclusion: Every ultrafilter F that converges to $x \in f^{-1}(U)$ contains $f^{-1}(U)$.

By Lemma ①, $f^{-1}(U)$ open. ▨

Proof of Tychonov Let $\{X_i\}_{i \in I}$ be a family of compact top. spaces. Want to show $X := \prod_{i \in I} X_i$ is compact, so let F be an ultrafilter thereupon.
Let $F_i := \pi_{i*} F$ be the pushforward of F on each X_i. By compactness on each X_i, F_i converges to at least one point in X_i.
Using Axiom of Choice, pick a limit x_i for each F_i.
So $x_i \in X_i$ and $F_i \to x_i$
Let $x := (x_i)_{i \in I} \in X$
Want to show that $F \to x$. Let U be an open nbhd of x. Then U contains a basic open nbhd B of x.

$$B = \bigcap_{k=1}^{n} \pi_{i_k}^{-1}(U_{i_k})$$

for open $U_{i_k} \subseteq X_i$ and $n \in \mathbb{N}_{\geq 1}$.

$\forall k = 1, \dots, n$ we know $U_{i_k} \ni x_i$ and U_{i_k} is open. So by the fact that $F_{i_k} \to x_{i_k}$ we get $U_{i_k} \in F_{i_k}$. We get $\pi_{i_k}^{-1}(U_{i_k}) \in F$
By closure under finite intersections of F, $B \in F$. By upwards closure $B \subseteq U \Rightarrow U \in F$, so every open nbhd of x is in F, so $F \to x$, so every ultrafilter converges to some point, leaving X compact. ▨

Throughout, X top. space.
(1) Let Y be a connected subspace of X. Prove that, if (A, B) is a separation of X, then either $Y \subseteq A$ or $Y \subseteq B$.
(partition into nonempty open sets)

$A \cap B = \emptyset$

Proof Let A, B be open, nonempty with $A \cup B = X$. Consider $Y \cap A$ and $Y \cap B$ open (def. subspace topology). If both of them were nonempty, then $(Y \cap A) \cup (Y \cap B) = Y \cap (A \cup B) = Y \cap X = Y$ would be a partition into nonempty open sets ↯

disjoint: $(Y \cap A) \cap (Y \cap B)$
$= Y \cap (A \cap B)$
$= Y \cap \emptyset = \emptyset$

So either $Y \cap A = \emptyset$ or $Y \cap B = \emptyset$. Equivalent:
$\Leftrightarrow Y \cap B^c = \emptyset$ $\quad \Leftrightarrow Y \cap A^c = \emptyset$
$\Leftrightarrow Y \setminus B = \emptyset$ $\quad \Leftrightarrow Y \setminus A = \emptyset$
$\Leftrightarrow Y \subseteq B$ $\quad \Leftrightarrow Y \subseteq A$
▨

(2) Let $\mathcal{C}$ be a collection of connected subspaces of X. Prove that, if $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$, then $\bigcup_{C\in\mathcal{C}} C$ is connected.

Proof Contraposition, so to prove if $\bigcup_{C\in\mathcal{C}} C$ is disconnected, then $\bigcap_{C\in\mathcal{C}} \mathcal{C} = \emptyset$.

Let (A,B) be a separation of $Y := \bigcup_{C\in\mathcal{C}} C$. Since for every $C \in \mathcal{C}$ C is connected and a subspace of Y, use (1) to obtain that either $C \subseteq A$ or $C \subseteq B$. There have to exist $C_1, C_2 \in \mathcal{C}$ s.t. $C_1 \subseteq A$ and $C_2 \subseteq B$ since if there were only subsets of B (resp. B), then $Y = \bigcup_{C\in\mathcal{C}} C \subseteq \bigcup_{C\in\mathcal{C}} A = A$ ↯ since $B \neq \emptyset$ (same argument for B)

Now $\bigcap_{C\in\mathcal{C}} C \subseteq C_1 \cap C_2 \subseteq A \cap B = \emptyset$ ▨

(3) Let A be a connected subspace of X. Prove that, if $A \subseteq B \subseteq \overline{A}$, then B is connected.

Proof Assume the opposite, so B is disconnected. Take a separation (E,F) of B. Now $A \subseteq B$ connected, so use (1) to obtain $A \subseteq E$ or $A \subseteq F$. $\Rightarrow \overline{A}^X \subseteq \overline{E}^X$ or $\overline{A}^X \subseteq \overline{F}^X$

$\Downarrow$

$F \subseteq B = E \cup F \subseteq \overline{A}^X \subseteq \overline{E}^X$

$(E_X \cup F_X) \cap B = (E_X \cap B) \cup (F_X \cap B) \subseteq \overline{E_X \cap B}^X$

$F \subseteq \overline{E}^X \setminus E$

E, F open in B

$\Rightarrow \exists E_X, F_X$: $E_X \cap B = E$

open in X $\quad F_X \cap B = F$

$E_X \cup F_X$

$\Rightarrow A \subseteq E$:

Since $F \neq \emptyset$, pick $x \in F$. Since F is open with $F \cap A = \emptyset$. Thus $x \notin A$ and x is not a limit point of A, so $x \notin \overline{A} \supseteq B \supseteq F$ ↯ ▨

(4) Base case: X_1, X_2 connected $\Rightarrow X_1 \times X_2$ connected

$A := X_1 \times \{x_2\}$ for $x_2 \in X_2$ connected $(\cong X_1)$,

$B_{x_1} := \{x_1\} \times X_2$ for $x_1 \in X_1$ connected.

Since $(x_1, x_2) \in A \cap B_{x_1} \overset{(2)}{\Rightarrow} A \cup B_{x_1}$ connected for every $x_1 \in X_1$

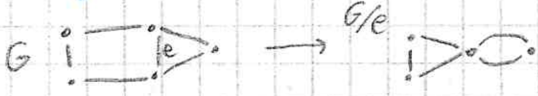
Now $\bigcap_{x_1 \in X_1} A \cup B_{x_1} \supseteq A \neq \emptyset \Rightarrow \bigcup_{x_1\in X_1} A \cup B_{x_1} = A \cup \bigcup_{x_1 \in X_1} \{x_1\} \times X_2$

$= X_1 \times X_2$ connected ▨

Graph minors

Throughout, $G = (V, E)$ is a graph.

For $e = uv \in E$, we let G/e denote the graph obtained from G by deleting the edge e and identifying the endpoints u and v. This process is called contraction of e.



Remark It is easily verified that for distinct edges $e, f \in E$ $G/e/f = G/f/e$. For this reason, given a subset $A \subseteq E$ of edges of G, we denote by G/A the graph obtained by successively contracting the edges of A.

Def.: For a graph G, a minor of G is a graph obtained from G by any sequence of the following operations:

(i) deleting an edge
(ii) contracting an edge
(iii) deleting an isolated vertex

(Order of operations is unimportant, since for disjoint edge-subsets $A, B \subseteq E$, $G/A\backslash B = G\backslash B/A$)

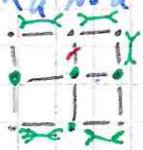
Def.: Minor relation

We say that a graph H is a minor of a graph G if it is isomorphic to a minor of G. In this case, write $H \preceq G$ and say that G contains an H-minor.

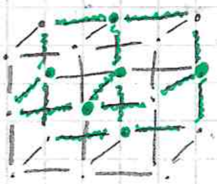
Examples (i)  is a minor of any graph containing a cycle

(ii)  $\cong K_3$ is a minor of any simple graph containing a cycle

(iii) K_4 is a minor of the (3×3)-grid



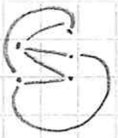
(iv) $K_{3,3}$ is a minor of the $(3\times3\times2)$-grid



Plane embeddings

Def.: Let $G=(V,E)$ be a graph without loops. An embedding of G in the plane is a map ι on $V \cup E$ s.t.:

(i) $\iota(v) \in \mathbb{R}^2 \quad \forall v \in V$
(ii) $\forall e \in E \quad \iota(e)$ is a piecewise linear arc in $\mathbb{R}^2$
(iii) ι is injective on V
(iv) a vertex v is never mapped to an interior point of $\iota(e)$ for any e
(v) a vertex v is incident to an edge e of G iff. $\iota(v) \in \iota(e)$
(vi) $\forall e, f \in E$ distinct, the interiors of $\iota(e)$ and $\iota(f)$ are disjoint.



Def.: A graph which has an embedding in the plane is called planar. A plane graph is a graph which "is embedded" in the plane – formally a pair (G, ι) where G is a graph and ι is an embedding of G.

Prop. Let $\overset{=(V,E)}{G}$ be a planar graph. Then every minor of G is planar.

Proof It suffices to show that $\forall e \in E$: $G\backslash e$ and G/e are planar. Fix an embedding ι of G in $\mathbb{R}^2$. Clearly the restriction of ι to $V \cup (E\backslash\{e\})$ is an embedding of $G\backslash e$, and so $G\backslash e$ is planar.

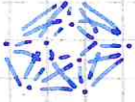
Henceforth, we may assume that e is not a loop (otherwise $G/e = G\backslash e$). Let x,y be the endpoints
Recall that the quotient space $\mathbb{R}^2/_{\iota(e)}$ is homeomorphic to $\mathbb{R}^2$.
So it suffices to prove that G/e has an embedding in $\mathbb{R}^2/_{\iota(e)}$
Define ι' as follows:
- $\forall v \in V\backslash\{x,y\}$: $\iota'(v) := \iota(v)$
- let $\iota'(z) = \iota(e)$, where z is the new vertex in G/e
- $\forall f \in E\backslash\{e\}$: $\iota'(f) := \iota(f)$

Then ι' is an embedding of G/e in $\mathbb{R}^2/_{\iota(e)} \cong \mathbb{R}^2$ ▨

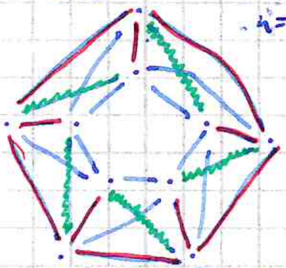
Def.: A face of a plane graph (G, ι) is a connected component of the topological space $\mathbb{R}^2 \backslash \iota(G)$.

→ equivalence classes under $x \sim y \Leftrightarrow \exists Y$ connected subspace : $x,y \in$

1. Find largest $n \in \mathbb{N}$ s.t.
$K_n \preceq$



$n=5$ ✓



$n=6$ also

$n \geq 8$ not possible since K_n would have at least $\frac{7 \cdot 8}{2} = 28$ edges, whereas we only have 25

$n \geq 7$ not possible because we can't increase degree of vertices

2. a) Show that $|E| \leq 3|V| - 6$ if G simple and has ≥ 3 vtx
no loops or double edges

Euler's formula: $v + f - e = 2 \Rightarrow e = v + f - 2$
$2e \geq 3f$ every face needs at least 3 edges, and every edge can contribute to two faces

$\rightsquigarrow f = e - v + 2 \quad | \cdot 3$
$3e - 3v + 6 = 3f \leq 2e \quad | -2e - 6 + 3v$
$e \leq 3v - 6$ ✓

b) Is K_5 planar? No, because otherwise it would fulfill the claim of a), but K_5 has 10 edges and 5 vertices
$10 \leq 3 \cdot 5 - 6 = 9$ ↯

c) Show that $|E| \leq 2|V| - 4$ if G is bipartite and has ≥ 4 vtx
can be partitioned into two parts with no edges within the parts
$\Rightarrow e \geq 2f$
$2e \geq 4f$ (no 3-cycles possible)
$\rightsquigarrow f = e - v + 2 \quad | \cdot 2$
$2e - 2v + 4 = 2f \leq e \quad | -e + 2v - 4$
$e \leq 2v - 4$

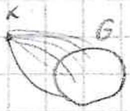
$K_{3,3}$ not planar, since it gives with $e = 9$ and $v = 6$
$\Rightarrow 9 \leq 2 \cdot 6 - 4 = 8$ ↯

3. Kuratowski's theorem: G planar $\Leftrightarrow K_5 \not\preceq G$ and $K_{3,3} \not\preceq G$
Def.: A graph G is outerplanar if $\exists$ embedding ι of G in $\mathbb{R}^2$ and a face f of ι s.t. every vertex of G lies on the boundary of f.

K_4 ✓

$K_{3,2}$ ✓

Prove: G outerplanar $\Leftrightarrow K_4 \not\preceq G$ and $K_{2,3} \not\preceq G$



Create $\hat{G}$ with a new vertex x which is connected to all existing vertices.
Claim: G outerplanar $\Leftrightarrow \hat{G}$ planar

Proof of Thm: G outerplanar $\overset{}{\underset{\text{Claim}}{\Leftrightarrow}} \hat{G}$ planar $\overset{\text{Kuratowski}}{\Leftrightarrow} K_5 \not\preceq \hat{G}$ and $K_{3,3} \not\preceq \hat{G}$

$\underset{(*)}{\Leftrightarrow} K_4 \not\preceq G$ and $K_{2,3} \not\preceq G$

Proof of Claim: [⇒] newly added edges lie on the face
[⇐]

(*) :)

Theorem Kuratowski's theorem (1930)

Graph G is planar $\Leftrightarrow$ $K_5 \not\preccurlyeq G$ and $K_{3,3} \not\preccurlyeq G$

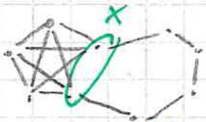
Proof of [$\Rightarrow$] Let G be planar, suppose for a contradiction that $K_5 \preccurlyeq G$ or $K_{3,3} \preccurlyeq G$.
Minors of planar graphs are planar $\Rightarrow$ K_5 is planar or $K_{3,3}$ is planar ↯ ▨

We prove [$\Leftarrow$] for the class of 3-connected graphs.

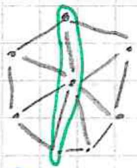
Prop. If G is 3-connected and $K_5 \not\preccurlyeq G$ and $K_{3,3} \not\preccurlyeq G$, then G is planar.

Def.: A graph G is k-connected if $|V(G)| > k$ and no separator of order $< k$ exists.
A separator of order l is a set $X \subseteq V(G)$, $|X| = l$ s.t. $G \setminus X$ is disconnected

separator of order 1
(1-separator)
$\Rightarrow$ Graph is not 2-connected



2-connected,
but not 3-connected



3-connected,
but not 4-connected

Lemma 1 If G is plane and 2-connected, then every face is bounded by a cycle.

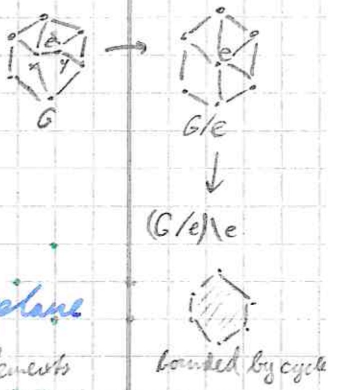
Proof idea:

If f not bounded by a cycle, we find a 1-sep. $\Rightarrow$ not 2-connected

Lemma 2 If $G \neq K_4$ and is 3-connected, $\exists e \in E(G)$ s.t. G/e is 3-connected
(Diestel Thm. 3.2.1.) $\Rightarrow$ allows induction on 3-connected graphs

Proof of Prop. Apply induction on $|V(G)|$.
Start $|V(G)| = 4$. Then $G = K_4$ which is planar.

Step Assume $|V(G)| > 4$ and that the prop. holds for smaller graphs.
By Lemma 2, $\exists e \in E(G)$ s.t. G/e is 3-connected. $e = \{x, y\}$
Since $K_5, K_{3,3} \not\preccurlyeq G$, $K_5, K_{3,3} \not\preccurlyeq G/e$ holds (contrapose).



Thus, G/e is planar by induction hypothesis.
Therefore, $G' := (G/e) \setminus e$ is still planar and 2-connected.
Let ι be an embedding of G/e in the plane. Let f be the face of G' containing $\iota(e)$.
Let C be the boundary of f in the embedding of G'.
By Lemma 1, C is a cycle.

Task: From G/e, uncontract e and either embed G in the plane or find a K_5 or $K_{3,3}$-minor of G.

$X :=$ set of neighbours of x on C
$Y :=$ —"— of y on C

even at least two elements

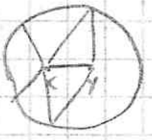
$X, Y \neq \emptyset$ because otherwise delete the other one x, y to make graph disconnected ↯ G 3-connected

Case 1 $Y \not\subseteq X$. Then let $y' \in Y \setminus X$. Let S be the minimum segment of C containing y' and has both end vertices in X.

Case 1(a) If $Y \subseteq S$, we can embed G in $\mathbb{R}^2$
Case 1(b) If $Y \not\subseteq S$, then $\exists y'' \in Y \setminus S$.
Then we get a $K_{3,3}$ minor

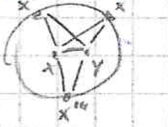
Case 2: $Y \subseteq X$

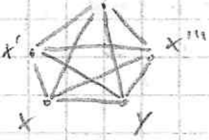
Case 2a Y contains two non consecutive elements of X on C
Then we find a $K_{3,3}$ minor



Case 2b $|Y| \le 2$ and contains two consecutive elements of X on C
Then we can embed this in the plane.
(not greater, because that would be case 2a)

Case 2c $|Y| = 3 = |X| \Rightarrow X = Y$
Then $K_5 \preceq G$





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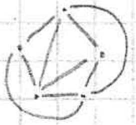


Remark General Kuratowski can be reduced to the 3-connected case.

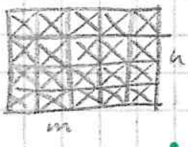
Alternative formulation of Kuratowski:
The set of excluded minors of the class of planar graphs is $\{K_5, K_{3,3}\}$.

Def: Given a minor-closed class $\mathcal{G}$ of graphs, then an excluded minor of $\mathcal{G}$ is a graph $H \notin \mathcal{G}$, but every proper minor of H is in $\mathcal{G}$.

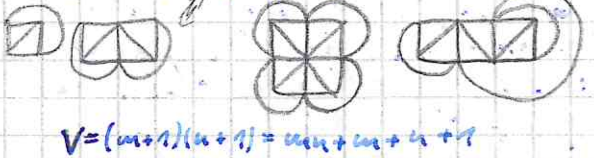
Research questions (1) What is the set $\mathcal{L}$ of excluded minors of graphs embeddable on the torus?
Known: $17523 \le |\mathcal{L}| < \infty$

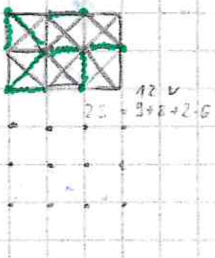
Theorem Robertson, Seymour (2004)
For any minor-closed class of graphs, the set of excluded minors is finite.

For which $(m,n) \in \mathbb{N}_{>0} \times \mathbb{N}_{>0}$ is the graph

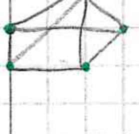


planar? (1,1), (1,2), (2,1), (2,2) (1,3)



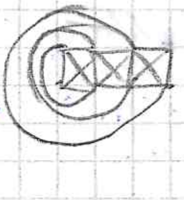


$V = (m+1)(n+1) = mn + m + n + 1$
$E = m(n+1) + n(m+1) + 2mn = mn + m + mn + n + 2mn$
$= 4mn + m + n$



$4mn + m + n \le 3mn + 3m + 3n - 3 \quad | -3mn - m - n$
$mn \le 2m + 2n - 3$ (4,3) non-planar
for $m, n \ge 3$ and one of them > 3: Show that $mn > 2m + 2n - 3$
IStart: $m = 4$, then
$4n \le 2n + 5 \quad | -2n$
$2n \le 5 \quad | :2$
$n \le \frac{5}{2}$. ↯
~~Istep: $(m+1)n = mn + n \overset{IH}{\le} 2m + 3n - 3$~~
~~$= 2(m+1) + 2n - 3 + (n-2)$~~
$(m+1)n = mn + n \overset{IH}{>} 2m + 3n - 3 = 2(m+1) + 2n - 3 + (n-2)$
$\ge 2(m+1) + 2n - 3$ ☑

K_5-minor



Prove that every graph embeds in R^3.
→ Put points at random positions, then probability for crossing lines is zero.

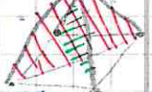
An embedding of a graph in R^3 is called **linked** if ∃ two vertex-disjoint cycles that are embedded as linked cycles, otherwise **linkless**.

K_5 and $K_{3,3}$ have linkless embeddings since they do not have enough vertices to contain vertex-disjoint cycles. → linkless graphs are a proper superclass of planar graphs.

Embedding in $\mathbb{R}^3$

A **2-dimensional simplicial complex** (2-complex) is a (simple) graph together with a set of triangles.
The **geometric realization** of a 2-complex (V,E,F) is the top. space obtained by starting with
- V with the discrete topology
- for every $(v,w) \in E$ we join v, w by an interval
- for every triangle $\Delta \in F$ we glue a 2-dim. disk with boundary Δ

An **embedding** of a 2-complex C into $\mathbb{R}^3$ is an injective continuous map from the geometric realisation of C into $\mathbb{R}^3$.

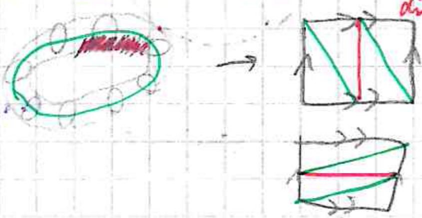


not an embedding since faces cross, but cone over  embeds

T top. space, the **cone over T** is $T \times [0,1] /_{T \times \{0\}}$
Example: G graph ⇒ cone over G is 2-complex
Its vtx set is $V(G) \cup t$ top of cone
edge set is $E(G) \cup \{xt \mid x \in V(G)\}$
face set is $\{\{x,y,t\} \mid x,y \in V(G),\ xy \in E(G)\}$

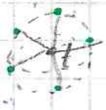
$X = K_5$ + all triangles as faces
Does X embed in $\mathbb{R}^3$? ⇒ yes (two tetrahedrons)
Given a 2-complex $C = (V,E,F)$, its **1-skeleton** is the graph (V,E).

Is the möbius strip with a disc glued onto the middle line embeddable?
Suppose for a contradiction $Y \hookrightarrow \mathbb{R}^3$.
Then take a toroidal nbhd around the central circle



disc intersects torus
Möbius strip intersects torus } intersect on torus
⇒ embedding not injective ↯

Lemma cone over K_5 does not embed in $\mathbb{R}^3$
Proof suppose for a contradiction $C(K_5) \hookrightarrow \mathbb{R}^3$.
Let t denote the top, take small sphere around t



all edges tx of $C(K_5)$ intersect the sphere in a poi

all faces txy intersect the sphere in the edge xy.
$C(K_5)$ intersects the sphere in K_5
⇒ $K_5 \hookrightarrow S^2$ ↯

▨

non-embeddable: start with cone over W_4 =  add two faces.

⤳ There is a Kuratowski type theorem for embeddability of simply connected 2-complexes in $\mathbb{R}^3$.

$\exists$ a linear time algorithm that checks whether a graph is planar!

A top. space X is simply connected if $\forall \gamma : \overset{\text{unit circle}}{S^1} \to X$ continuous
$\exists \hat{\gamma} : S^1 \times [0,1] \to X$ continuous s.t. $\hat{\gamma}|_{S^1 \times \{0\}} = \gamma$,
$\hat{\gamma}|_{S^1 \times \{1\}} = \text{const.}$

1-dimensional simplicial complexes (geom. realisations of graphs):

K_1 is simply connected because $\exists!\ \gamma : S^1 \to K_1$.

K_2: Let $\gamma : S^1 \to \underset{\cong[0,1]}{K_2}$ be given

WLOG: $\gamma : S^1 \to [0,1]$
Now define for $t \in [0,1]$ $\gamma_t : S^1 \to [0,1]$
$\gamma_t(x) = (1-t) \cdot \gamma(x)$

$\hat{\gamma} : S^1 \times [0,1] \to K_2$
$(x,t) \mapsto \gamma_t(x)$ is continuous:
Let $U \subseteq [0,1]$ be open. Want to show $\hat{\gamma}^{-1}(U)$ open.
For this show that $\forall x \in \hat{\gamma}^{-1}(U)$ $\exists O_x$ open in $S^1 \times [0,1]$ with
$x \in O_x$, $O_x \subseteq \hat{\gamma}^{-1}(U)$

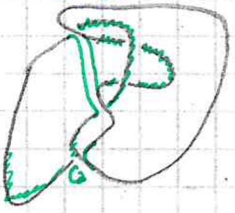
Let $(x,t) \in \hat{\gamma}^{-1}(U)$ be arbitrary. Let $y \in [0,1]$ with $\hat{\gamma}(x,t) = y$.
Find O_x' open in S^1 s.t. $x \in O_x'$, $\gamma_t(O_x') \subseteq U$

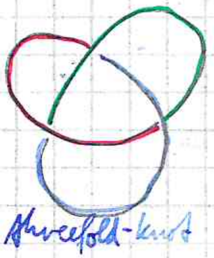
$$\hat{\gamma}^{-1}(U) = \{(x,t) \mid \hat{\gamma}(x,t) \in U\} = \{(x,t) \mid (1-t) \cdot \gamma(x) \in U\}$$
$$= \{(x,t) \mid \gamma(x) \in \tfrac{1}{1-t} \cdot U\}$$

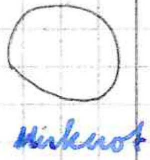
K_3 not simply connected
$\Rightarrow$ simply connected graphs are exactly the ones without any cycles

Fact It is undecidable whether a 2-complex is simply connected.

Fact For 2-complexes $\hookrightarrow \mathbb{R}^3$, checking simply connectedness is in NP.







Knots

Def.: A knot is a piecewise linear embedding of S^1 into $\mathbb{R}^3$.
Two knots K_1, K_2 are equivalent if $\exists$ a homeomorphism of $\mathbb{R}^3$ onto itself mapping K_1 to K_2.

Any knot equivalent to the unknot is called trivial.

Q: Is there a polynomial time algorithm that decides whether a knot is trivial?

A knot diagramme is a plane 4-regular multigraph together with an Euler tour s.t. at each vertex we pick one of the two traversals of the Euler tour to be "on top".

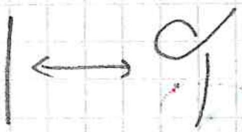
Given a knot diagramme, there is a natural way to construct a knot out of it:
- the Euler tour is a piecewise linear image of S^1.
- lift at each vertex the traversal "on top" a little bit.

⇒ associated knot

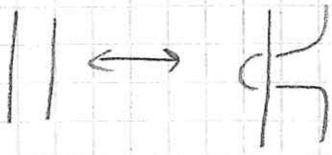
Thm ∀ knot K ∃ a knot diagramme so that its associated knot is equivalent to K.

Reidemeister moves

R1



R2

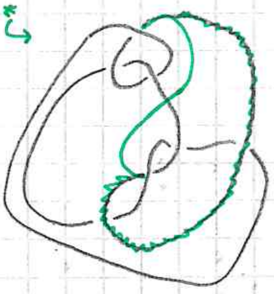


R3

Two knot diagrammes are Reidemeister-equivalent if ∃ sequence of Reidemeister moves transforming one to the other.

Thm Two knot diagrammes have equivalent associated knots iff they are Reidemeister-equivalent.

↳



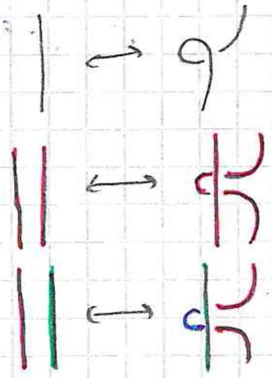
(somehow possible)

Find an invariant to prove that the threefold-knot is not trivial.
We say a knot-diagramme is 3-colourable if we can assign one of 3 colors to each of its segments s.t. at each crossing only one colour appears or all 3 & ≥2 colors are used.
without this: "weak"

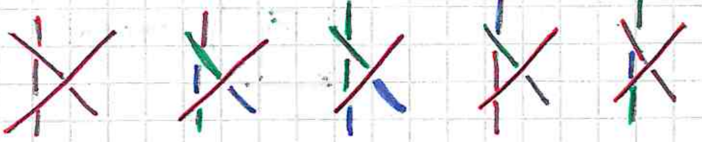
Thm Being 3-colourable is a property invariant under Reidemeister moves.

⇒ the threefold knot is nontrivial.

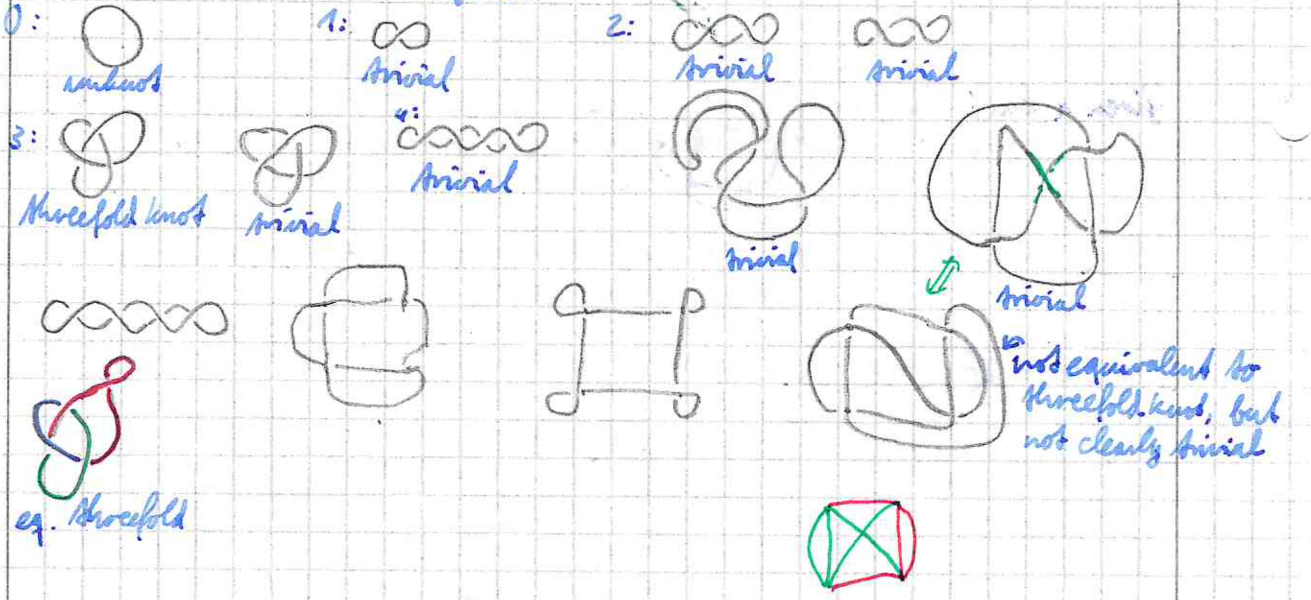
Proof. Look at each Reidemeister move separately.



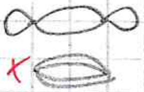
unique weak 3-colourings



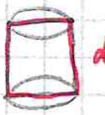
Problem: Classify all knots with at most 4 crossings.
(diagrams)

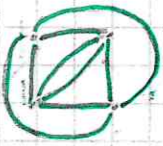


- unknot 
- on ≤ 2 crossings, no 3-colourable knots

crossings	graphs	knots
0 1	∅ 	
2		
3	 ≥ 1 loop  only graph	unknot –"– ~~8~~ 4 threefold  trivial
4	≥ 1 loop ⇒	reduce to ≤3 crossings


disconnected


disconnected


connected

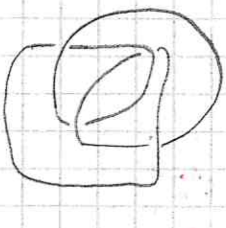


Figure-8-knot

Fundamental groups

X top. space, $x_0 \in X$.

$$\pi_1(X, x_0) := \underbrace{\{ f : \underset{\overset{\shortparallel}{I}}{[0,1]} \to X \overset{\text{continuous}}{,} \ f(0) = f(1) = x_0 \}}_{F_1(X, x_0)} /\!\sim$$

← two curves equivalent if they can be continuously moved to another

Lemma $\overset{\pi}{\cancel{F}}_1(X, x_0)$ is a group under concatenation.

Proof
- closed under concatenation ✓
- neutral element $\begin{aligned} f: I &\to X \\ x &\mapsto x_0 \end{aligned}$
- associativity: $(f \circ g) \circ h = f \circ (g \circ h)$
- existence of inverses

  Given $f : I \to X$ continuous, inverse should go the other way

  → we have to introduce eq. relation ~

Def. Given $f, g : I \to X$, $^{x_0 \in X}$ a homotopy from f to g is a continuous map $h: I^2 \to X$ s.t. $\quad (x,t) \in I^2 \mapsto h(x,t)$

$$h|_{t=0} = f \qquad h(0,t) = x_0$$
$$h|_{t=1} = g \qquad = h(1,t)$$

Ex. $\vec{f} \circ \overleftarrow{f}$ is homotopically equivalent $^{\exists \text{ homotopy}}$ to the constant function $c: I \to f(0)$.

$$\begin{aligned} \vec{f}: I &\to X \\ x &\mapsto f(x) \end{aligned} \qquad \begin{aligned} \overleftarrow{f}: I &\to X \\ x &\mapsto f(1-x) \end{aligned}$$

$$(f \circ g)(x) = \begin{cases} x \leq \frac{1}{2}: & f(2x) \\ x \geq \frac{1}{2}: & g(2(x - \frac{1}{2})) \end{cases}$$

only well-def. for $f(1) = g(0)$

$$(\vec{f} \circ \overleftarrow{f})(x) = \begin{cases} x \leq \frac{1}{2}: & \vec{f}(2x) \\ x \geq \frac{1}{2}: & \overleftarrow{f}(2(x - \frac{1}{2})) \end{cases} = \begin{cases} x \leq \frac{1}{2}: & f(2x) \\ x \geq \frac{1}{2}: & f(1 - 2(x - \frac{1}{2})) = f(2 - 2x) \end{cases}$$

Prop: Homotopy equivalence is an equivalence relation.

Proof: reflexive: constant homotopy $h(x,t) = f(x)$
$f \sim f$

symmetry: $f \sim g \Rightarrow g \sim f$: Take $\hat{h}(x,t) := h(x, 1-t)$

transitivity: $f \sim g$, $g \sim h \Rightarrow f \sim h$: concatenate w.r.t. time argument ∎

Lemma $\pi_1(X, x_0)$ is a group under concatenation. $\quad [f] \circ [g] := [f \circ g]$
← well-defined?

Proof:
- closed under concatenation ✓
- neutral element: $[f]$ with $\begin{aligned} f: I &\to X \\ x &\mapsto x_0 \end{aligned}$
- associativity $(f \circ g) \circ h = f \circ (g \circ h)$
- existence of inverses

  $[\vec{f}] \circ [\overleftarrow{f}] = [\vec{f} \circ \overleftarrow{f}] \overset{\text{example}}{=} [\text{const.}]$ ∎

Fact: If X is path-connected, then $\forall x_0, x_1 \in X$: $\pi_1(X, x_0) \cong \pi_1(X, x_1)$
We can therefore write $\pi_1(X)$.

Functor

top. space → group

$X \mapsto \pi_1(X)$

If $\pi_1(X) \not\cong \pi_1(Y)$, then X is not homeomorphic to Y.

Equivalent: If X and Y are homeomorphic, $\pi_1(X)$ and $\pi_2(Y)$ are isomorphic.

Knot complements

Given a knot $K \hookrightarrow \mathbb{R}^3$ a knot complement of K is obtained from $\mathbb{R}^3$ by removing a small toridal neighbourhood around K.

Theorem Gordon, Luecke 1989

Two knots are equivalent iff their knot complements are homeomorphic.

Corollary If the fundamental group of the knot complement of K is not $\mathbb{Z}$, then K is a non-trivial knot. $\square$

Plan: Give description of fundamental group of knot complement in terms of knot diagram
- Method to show that a group is not $\mathbb{Z}$
- Construct nontrivial group homomorphism to the dihedral group

What are the fundamental groups of these top. spaces?

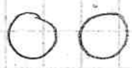
Ⓐ S^1

$\mathbb{Z}$ ... integer number of windings

Ⓑ $S^1 \vee S^1$

$\mathbb{Z} * \mathbb{Z}$ free product (words of letters where aa^{-1} and bb^{-1} cancel out)

Ⓒ $S^1 \cup S^1$



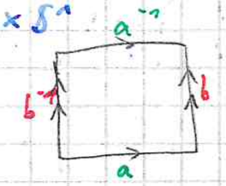
not path-connected, so for each base point $\mathbb{Z}$

Ⓓ

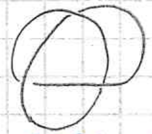
$\mathbb{Z} \times \mathbb{Z}$

torus

$S^1 \times S^1$



Ⓔ



$\cong$

$\mathbb{Z}$

trefoil knot

If a group enters the room it usually doesn't say "Hey, here's my multiplication table"

Presentation of groups:

S set, W words in S, R set of words in S.

$w_1, w_2 \in W$ are related if $w_1 = w_2^{-1}$ or $w_1 w_2^{-1} \in R$ or $w_2 w_1^{-1} \in R$

$$\overset{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\parallel} \qquad \overset{x_n^{-\alpha_n} \cdots x_1^{-\alpha_1}}{\parallel}$$

Take transitive hull of being related (this is an equivalence relation). Let G be the set of those equivalence classes.

Prop.: G with composition of words is a group.

Examples: ① $S = \{a\}$, $R = \emptyset$

Words: a^z, $z \in \mathbb{Z}$

$G \cong \mathbb{Z}$

② $S = \{a\}$, $R = \{a^3\}$

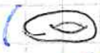
Words: a^z $z \in \mathbb{Z}$

$a^7 \sim a^4$ as $a^7 a^{-4} = a^3 \in R$

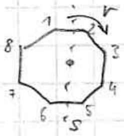
$a^{10} \sim a^4$ as $a^{10} a^{-4} = a^6 \in R$ but $a^{10} \sim a^7 \sim a^4 \Rightarrow a^{10} \overset{\text{trans.}}{\sim} a^4$

eq. classes: $\{a^{3k}, k \in \mathbb{Z}\}$, $\{a^{3k+1}\}$, $\{a^{3k+2}\}$
$G \cong \mathbb{Z}_3$

③ $S = \{a, b\}$, $R = \emptyset$
$\dashrightarrow$ $G = \mathbb{Z} * \mathbb{Z}$ free product

④ $S = \{a, b\}$ $R = \{aba^{-1}b^{-1}\}$
$G = \mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z} = \{a^i b^j \mid i, j \in \mathbb{Z}\} = \pi_1($  $)$

⑤ Dihedral groups:
regular n-gon in $\mathbb{R}^2$



$S = \{\overset{\text{rotate by } \frac{2\pi}{n}}{r}, \underset{\text{fixed reflection}}{s}\}$

$R = \{r^n, s^2, sr^k sr^k\}$

$srs =$  $= r^{-1}$

Elements of the dihedral group have form $\underset{\text{rotation}}{\underset{\uparrow}{r^k}}$ or $\underset{\text{reflection}}{\underset{\uparrow}{sr^k}}$

No two reflections commute

Proof of Prop.: neutral element: $[\varepsilon]^{\ni aa^{-1}}$
inverse elements: $[x_1^{\alpha_1} \cdots x_n^{\alpha_n}]^{-1} = [x_n^{-\alpha_n} \cdots x_1^{-\alpha_1}]$

well-defined: $\overset{\Leftrightarrow w_1 w_2^{-1} \in R}{w_1 \sim w_2}$, $\overset{\Leftrightarrow u_1 u_2^{-1} \in R}{u_1 \sim u_2}$.
Show $w_1 u_1 \sim_{trans} w_2 u_2$

Show
$w_1 u_1 \underset{\uparrow}{\sim} w_1 u_2 \sim w_2 u_2$

$w_1 u_1 (w_1 u_2)^{-1} = w_1 \underbrace{u_1 u_2^{-1}}_{\in R} w_1^{-1} \in R$

$w_1 u_2 (w_2 u_2)^{-1} = w_1 u_2 u_2^{-1} w_2^{-1} = w_1 w_2^{-1} \in R$

▨

Prop. Given $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$ presented groups. Then every function $f: S_1 \to S_2$ extends to a group homomorphism φ via

$$\begin{tikzcd} G_1 = \langle S_1 \mid R_1 \rangle \arrow[d, "f"'] \\ G_2 = \langle S_2 \mid R_2 \rangle \end{tikzcd}$$

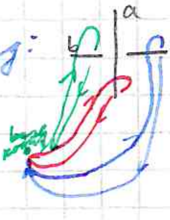
$\varphi(x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}) := f(x_1)^{\alpha_1} \cdots f(x_n)^{\alpha_n}$

if $\forall\ y_1^{\beta_1}, \dots, y_m^{\beta_m} \in R_1$ we have
$f(y_1)^{\beta_1}, \dots, f(y_m)^{\beta_m} = 1_{G_2}$

Corollary 5 Let G be a presented group with a group homomorphism induced via a function f as above into a dihedral group so that two reflections are in the image of f. Then G is not Abelian. In particular: $G \ncong \mathbb{Z}$.

arcs: segments of knot diagramm from one undercrossing to the next

Theorem Given a knot diagramme D of a knot K, the fundamental group of the knot diagramme has the following presentation:

generators: arcs
one relation for each crossing:  : $b = aca^{-1}$

Try to move b-loop to c-loop, but a is in the wa

Define a map f mapping each arc to a reflection in the dihedral group
First investigate at a single crossing:

$a \longmapsto sr^{\alpha}$
$b \longmapsto sr^{\beta}$
$c \longmapsto sr^{\gamma}$

We need $sr^{\beta} \overset{!}{=} \underbrace{sr^{\alpha}s}_{r^{-\alpha}}r^{\gamma}\underbrace{(sr^{\alpha})^{-1}}_{r^{-\alpha}s}$

$= r^{-\alpha}r^{\gamma}r^{-\alpha}s \quad | \cdot s^{-1}$

$r^{-\beta} = r^{-\alpha}r^{\gamma}r^{-\alpha}$
$\Rightarrow \quad 1 = r^{\beta+\gamma-2\alpha} \qquad (*)$

General FOX-coloring:

Let D be a knot diagramme of a knot K. Assume the arcs of D can be colored with elements of $\mathbb{Z}_n$ s.t. ≥ 2 colors are used & at every crossing $b|^{\alpha}_{-c}$ we have $b + c \equiv 2a \mod n$

Then K is a non-trivial knot.

Proof: To show that K is non-trivial, it suffices to show by Cor. 5 to def. a function f: arcs $\to$ dihedral group of order $2n$ s.t. $|\text{Im } f| \geq 2$ and all images are reflections.

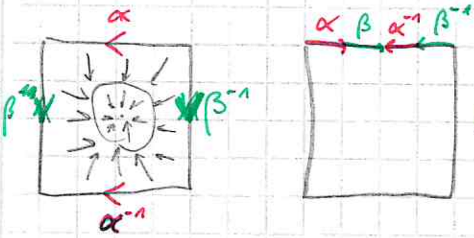
s.t. $\forall$ crossing $b = aca^{-1}$. True by ⊛ ▨

knot diagramme D $\xrightarrow{\text{Gordon Luecke}}$ knot complement $\to$ π_1 representation by arcs of D
$\downarrow$ group theory
Cor 5 to show $\pi_1 \neq \mathbb{Z}$

Figure 8-knot

mod 5

This is how a topologist proves it: *malt zwei Striche an die Tafel*



homotopy

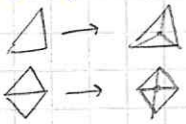
Manifolds

Def.: A (topological) manifold of dimension d is a topological space M s.t. $\forall x \in M$ $\exists$ a nbhd $U = U(x)$ and a homeomorphism $\varphi_x : U_x \to \mathbb{R}^d$
$\underset{B_1^d(0)}{\cong}$

A surface is a compact 2-dimensional connected manifold.
A [illegible] 2-dim. simplicial complex c is called a triangulation of a surface S if the geometric realisation of c is homeomorphic to S.

Facts:
Every surface has a triangulation.

Def.: A triangulation T is a refinement of a triangulation U if T can be obtained from U by applying subdivisions:



Fact: For any two triangulations of a surface S there ex. a triangulation refining both.

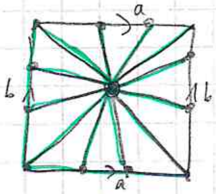
The Euler characteristic of a triangulation T is $v(T)-e(T)+f(T) =: \chi(T)$

where $v(T) = |V(T)|$

$e(T) = |E(T)|$

$f(T) = |F(T)|$

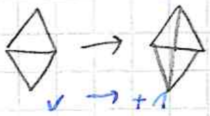
Ex.



$\chi(T) = 6 - 18 + 12 = 0$

Lemma A triangulation has Euler characteristic as any of its refinements

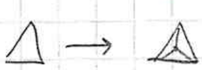
Proof: By induction.



$v \to +1$

$e \to +3$

$f \to +2$



$v \to +1$

$e \to +3$

$f \to +2$

▨

Corollary Any two triangulations of the same surface have the same Euler characteristic.

Proof: Let T, U be such triangulations. Then they have a common refinement R. By previous lemma (2x)

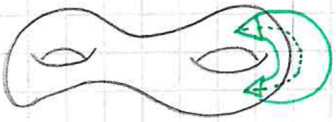
$\chi(T) = \chi(R) = \chi(U)$

▨

This motivates

Def.: The Euler-characteristic of a surface S is the Euler characteristic of any of its triangulations.

Two surfaces with different Euler characteristic are not isomorphic.

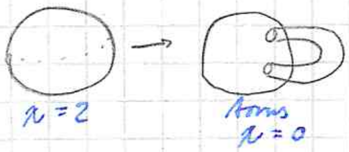


How does adding a handle change the Euler characteristic?

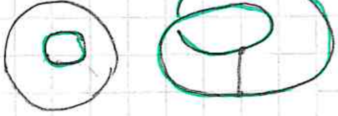
- lose 2 faces
- no new vertices
- +6 edges
- +6 faces

⇒ decreases Euler characteristic by 2

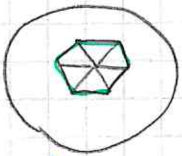
Ex.:

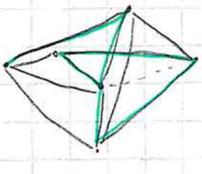


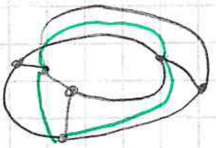
Start with S^2, take out a disc (boundary is S^1) and glue a möbius strip there.

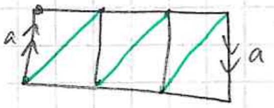


How does this operation affect Euler characteristic?









-1 v

-6 e

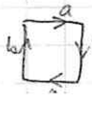
-6 f

+6 e

+6 f

$\Rightarrow -1$

Putting in this möbius strip decreases χ by 1.



≅ Klein Bottle

Thm: Classification of surfaces

Let S be a surface with Euler characteristic χ.
If $\chi = 2 \Rightarrow S \cong S^2$
$\chi = 1 \Rightarrow S \cong$ cross cap (disc + möbius strip)
$\chi = 0 \Rightarrow S \cong$ torus or $S \cong$ Klein bottle
$\chi = 2-(2g+1)$ odd, then $S \cong S^2$ with $2g+1$ Möbius strips weaved in
$\chi = 2-2g$ even $\Rightarrow S \cong \begin{cases} S^2 \text{ plus } g \text{ handles} \\ \text{or} \\ S^2 \text{ plus } 2g \text{ Möbius strips} \end{cases}$

Cor: Two surfaces containing a Möbius strip are isomorphic iff they have the same Euler characteristic.

Cor: -"- not containing a Möbius strip.