Problem 1.1. For $m, n \in \mathbb{N}$, let $U_1, \dots, U_m \subset \mathbb{R}^n$ be convex open subsets such that any three of them have non-empty intersection. Show that their union

$$X = \bigcup_{i=1}^{m} U_i \quad \text{is simply connected.}$$

Problem 1.2. Let Σ_g be a compact orientable surface of genus g.

(a) Let $\Sigma_{g,n} = \Sigma_g \setminus (\mathbb{D}_1 \cup \dots \cup \mathbb{D}_n)$ where $\mathbb{D}_1, \dots, \mathbb{D}_n$ are pairwise disjoint open disks. Show that

$$\pi_1(\Sigma_{g,n}, *) \quad \text{is a free group on } 2g + n - 1 \text{ generators.}$$

(b) Let $C \subset \Sigma_g$ be a circle that separates Σ_g into two disjoint pieces $\Sigma_{g_1,1}$ and $\Sigma_{g_2,1}$ as shown in the following picture:

[PICTURE]

Show that

- Σ_g does not retract[1] onto the separating circle C.
- Σ_g does retract onto the non-separating circle D.

Problem 1.3. The *Klein bottle* is defined to be the topological space

$$\mathbb{K} = [0,1]^2 / \sim$$

where $\sim$ denotes the identifications given by $(s, 0) \sim (s, 1)$ and $(0, t) \sim (0, 1-t)$ for all s, t.

(a) Show that $\pi_1(\mathbb{K}, *) \simeq \langle a, b \mid aba^{-1}b = e \rangle$.

(b) Show that $\ell = \{[s,t] \in \mathbb{K} \mid t = 1/4 \text{ or } t = 3/4\}$ is a closed curve that separates $\mathbb{K}$ in two copies of the *Möbius strip*

$$\mathbb{M} = \{(e^{i\theta}, \tau e^{i\theta/2}) \in S^1 \times \mathbb{C} \mid \theta \in [0, 2\pi], \tau \in [-1, 1]\}.$$

Deduce via the Seifert – van Kampen theorem that also $\pi_1(\mathbb{K}, *) \simeq \langle c, d \mid c^2 = d^2 \rangle$.

(c) The *real projective plane* is the quotient $\mathbb{RP}^2 = \mathbb{C}/\sim$ of the unit disk $\mathbb{D} \subset \mathbb{C}$ by the identification $z \sim -z$ for $z \in S^1 = \partial\mathbb{D}$. Show that the Klein bottle can also be written as a connected sum

$$\mathbb{K} \cong \mathbb{RP}^2 \# \mathbb{RP}^2$$

and deduce from this result yet another presentation $\pi_1(\mathbb{K}, *) \simeq \langle x, y \mid x^2 y^2 = e \rangle$.

(d) Find explicit isomorphisms between the above three presentations of the fundamental group. Hint: First determine the abelianization for each of the occuring groups.

[1] We say that a space X *retracts* onto $Y \subset X$ if there is a continuous map $r \colon X \to Y$ with $r|_Y = \mathrm{id}$.