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Algebraic Topology

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Chapter I
Introduction

In topology we study geometric shapes from a qualitative viewpoint, with a focus on features that are invariant under continuous deformation. In this context metric aspects such as distances or angles become secondary. Instead, we use the much more flexible notion of a topological space. These are made of rubber: We may bend, stretch or compress them continuously, as long as we don't tear them apart and don't glue pieces together. More precisely, the only data remembered by a topological space are its open subsets (one could say that these give a notion of two points being near to each other without talking about distances):

Definition. A *topology* on a set X is a collection $\mathscr{T}$ of subsets of X such that

a) $\mathscr{T}$ is stable under arbitrary unions,
b) $\mathscr{T}$ is stable under finite intersections, and
c) both X and the empty set $\varnothing$ belong to $\mathscr{T}$.

We then call $(X, \mathscr{T})$ a *topological space*. The sets in $\mathscr{T}$ are called the *open subsets* of the topological space. A subset $Z \subset X$ is called *closed* if $X \setminus Z$ is open.

Example. The following are topological spaces:

a) $X = \mathbb{R}^n$ with the Euclidean topology, where by definition a subset $U \subset X$ is open iff for every $p \in U$ there exists $\varepsilon > 0$ such that

$$B_\varepsilon(p) := \{q \in X \mid d(p,q) < \varepsilon\} \subset U.$$

Here $d(p,q) := |p-q|$ denotes the Euclidean distance between p and q.

b) If X is a topological space, then any subset $Y \subset X$ inherits a topology where we define a subset $V \subset Y$ to be open iff

$$V = U \cap Y \quad \text{for some open subset} \quad U \subset X.$$

We call this the *subspace topology* and $Y \subset X$ a *topological subspace* of X.

c) By combining these two examples, any subset $Y \subset \mathbb{R}^n$ has a natural topology.

d) For any topological spaces X and Y, the *product topology* on $Z = X \times Y$ is by definition the topology whose open subsets are unions

$$W = \bigcup_{i \in I} U_i \times V_i \subset Z = X \times Y$$

where $U_i \subset X$ and $V_i \subset Y$ are open and I is an arbitrary index set.

e) For any topological spaces X and Y we endow their *disjoint union* $Z = X \sqcup Y$ with the topology whose open subsets are the unions

$$W = U \sqcup V \quad \text{of open subsets} \quad U \subset X \quad \text{and} \quad V \subset Y.$$

f) Any topology on a set X sits between the following two extreme topologies:

- The *discrete topology* where every subset of X is defined to be open.
- The *indiscrete topology* whose only open sets are X and the empty set $\varnothing$.

When dealing with topological spaces, we are mainly interested in maps between them that are compatible with the respective topologies:

Definition. A map $f : X \to Y$ between topological spaces is *continuous* if for any open $V \subset Y$ the preimage $f^{-1}(V) \subset X$ is again open. We call f a *homeomorphism* if it is continuous, bijective and if the inverse $f^{-1} : Y \to X$ is also continuous. We then also write

$$X \cong Y$$

and say that X and Y are *homeomorphic*.

Example. Saying that two spaces are homeomorphic is a precise version of saying that they look 'the same' topologically. Here are some examples:

a) The unit circle is homeomorphic to a square. An explicit homeomorphism can be obtained by projecting points radially from the center:

[PICTURE]

b) The exponential function gives a continuous bijective map

$$f : \quad [0, 2\pi) \to S^1 := \{z \in \mathbb{C} \mid |z| = 1\}, \quad t \mapsto \exp(it),$$

but its inverse is not continuous. In fact the topological spaces $[0, 2\pi)$ and S^1 are not homeomorphic. A quick argument for this uses compactness: A topological space X is called *compact* if every cover $X = \bigcup_{i \in I} U_i$ by open sets $U_i \subset X$ admits a finite subcover, i.e. one can find a finite collection of indices $i_1, \dots, i_n \in I$ such that

$$X = U_{i_1} \cup \cdots \cup U_{i_n}$$

Since S^1 is compact while $[0, 2\pi)$ is not, they cannot be homeomorphic.

c) The unit interval $I = [0,1]$ is not homeomorphic to the square $Q = I \times I$. This is not as obvious as it seems: Both have the same cardinality and there even exist continuous surjective maps

$$f\colon \quad I \to Q = I \times I$$

known as space-filling curves (`en.wikipedia.org/wiki/Peano_curve`). A quick argument for the non-existence of homeomorphisms uses the notion of path-connectedness: A topological space X is called *path-connected* if for any points $p, q \in X$ there is a continuous map $f\colon [0,1] \to X$ with $f(0) = p$ and $f(1) = q$ as shown in the following picture:

[PICTURE]

If we remove an interior point from I, the remaining open subset is no longer path-connected. By way of contrast, in Q the complement of any point is still path-connected. Hence I and Q cannot be homeomorphic.

d) It seems obvious that the following two spaces are not homeomorphic:

[PICTURE]

Intuitively, the reason is that the former has only one hole while the latter has two holes. But how can we make this into a formal argument?

The basic idea of algebraic topology is to attach to any topological space algebraic invariants (groups, rings, vector spaces, ...) which encode properties such as ‘holes’ in a meaningful way. The constructions will be made so that any continuous map of topological spaces induces a homomorphism between their algebraic invariants, and so that homeomorphisms induce isomorphisms. In particular, if two spaces have non-isomorphic algebraic invariants, then the spaces cannot be homeomorphic. As a slogan:

Algebraic topology converts topology (hard) into algebra (easy)!

In this class we will see two main examples for this approach, homotopy groups and homology groups. They are not only good for showing that two topological spaces are not homeomorphic, but have plenty of other applications. For instance, we will soon be able to give very simple proofs of

- the fundamental theorem of algebra: Every non-constant polynomial over the complex numbers has a complex root,
- the Brouwer fixed point theorem: Every continuous map from a disk to itself has a fixed point,
- the Jordan curve theorem: Every simple closed curve in the plane separates the plane into an inside and an outside region,
- the Borsuk–Ulam theorem: There are always two opposite points on Earth with the same temperature and air pressure,
- ...

Chapter II
The fundamental group

1 Preliminaries about paths

Let X be a topological space. We want to study it in terms of paths:

Definition 1.1. A *path in X* is a continuous map $\gamma\colon [0,1] \to X$. To keep track of the end points $a = \gamma(0)$ and $b = \gamma(1)$, we also say γ is a path from a to b. The *inverse* of γ is defined by

$$\gamma^{-}\colon \quad [0,1] \to X, \quad s \mapsto \gamma(1-s).$$

A path $\mu\colon [0,1] \to X$ is said to be *composable with* $\gamma\colon [0,1] \to X$ if $\mu(0) = \gamma(1)$, in which case we put

$$\gamma \cdot \mu\colon \quad [0,1] \to X, \quad s \mapsto \begin{cases} \gamma(2s) & \text{if } 0 \le s < \frac{1}{2}, \\ \mu(2s-1) & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$

Note that we denote the concatenation of paths from left to right!

[PICTURE OF INVERSE AND COMPOSITE]

Lemma 1.2. *Define a binary relation $\sim$ on X by writing $a \sim b$ if there exists a path from a to b in X. Then $\sim$ is an equivalence relation.*

Proof. The relation is

- reflexive: For any $a \in X$ we have $a \sim a$ via the constant path $\gamma\colon [0,1] \to \{a\}$.
- symmetric: If $a \sim b$ via a path γ, then $b \sim a$ via γ^-.
- transitive: If $a \sim b$ via a path γ and $b \sim c$ via a path μ, then $a \sim c$ via $\gamma \cdot \mu$. □

The equivalence classes for the above relation $\sim$ are called the *path components* of X. We put

$$\pi_0(X) = \{\text{path-components of } X\}.$$

Note that this is just a set, with no extra structure. We say that X is *path-connected* if it has only one path-component, that is, if any two points $a, b \in X$ can be joined by a path in X. In general, any topological space X decomposes canonically as the disjoint union

$$X = \bigsqcup_{Y \in \pi_0(X)} Y$$

of its path components $Y \subset X$, and each of those is path-connected. So in discussing paths we may assume without loss of generality that X is path-connected.

2 Definition of the fundamental group

The notion of a path is too rigid to be useful. In practice we only care about paths up to continuous deformation:

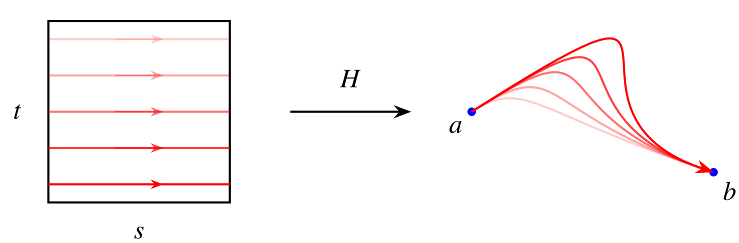
Definition 2.1. Let $\gamma_0, \gamma_1\colon [0,1] \to X$ be two paths in X. A *homotopy* between them is a continuous map

$$H\colon [0,1] \times [0,1] \to X \quad \text{with} \quad H(s,0) = \gamma_0(s) \quad \text{and} \quad H(s,1) = \gamma_1(s) \quad \text{for all } s.$$

If γ_0 and γ_1 are paths from a to b, we call H a *homotopy through paths from a to b* if moreover

$$H(0,t) = a \quad \text{and} \quad H(1,t) = b \quad \text{for all } t.$$

We then say that the two paths γ_0 and γ_1 are *equivalent* and write $\gamma_0 \sim \gamma_1$.



Lemma 2.2. *The relation ~ is an equivalence relation on the set of paths in X.*

Proof. The relation is

- reflexive: We have $\gamma \sim \gamma$ via the constant homotopy $H(s,t) := \gamma(s)$.
- symmetric: If $\gamma_0 \sim \gamma_1$ via some homotopy H, then also $\gamma_1 \sim \gamma_0$ via the inverse homotopy

$$H^-: \quad [0,1]\times[0,1] \to X \quad \text{defined by} \quad H^-(s,t) := H(s,1-t).$$

- transitive: If $\gamma_0 \sim \gamma_1$ via a homotopy H' and $\gamma_1 \sim \gamma_2$ via a homotopy H'', then

$$\gamma_0 \sim \gamma_2 \quad \text{via the homotopy} \quad H(s,t) := \begin{cases} H'(s,2t) & \text{for } 0 \le t < \frac{1}{2} \\ H''(s,1-2t) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Note that the map H defined by the above formula is continuous. □

We denote by $[\gamma]$ the equivalence class of a path γ for the equivalence relation in the above lemma. The inversion and composition of paths are well-defined on the level of these equivalence classes:

Lemma 2.3. *Let γ_0, γ_1 be two paths in X.*

a) If $\gamma_0 \sim \gamma_1$, then also their inverses satisfy $\gamma_0^- \sim \gamma_1^-$.
b) If μ_0, μ_1 are composable with γ_0, γ_1 and if $\mu_0 \sim \mu_1$, then also $\gamma_0 \cdot \mu_0 \sim \gamma_1 \cdot \mu_1$.

Proof. To prove $b)$, pick any homotopies $H': \gamma_0 \sim \gamma_1$ and $H'': \mu_0 \sim \mu_1$ between the respective paths. We can then define a homotopy between their composites via the formula

$$H(s,t) := \begin{cases} H'(2s,t) & \text{for } 0 \le 2 < \frac{1}{2}, \\ H''(2s-1,t) & \text{for } \frac{1}{2} \le 2 \le 1. \end{cases}$$

The proof of $a)$ is similar and left as an exercise. □

We now specialize to the most important case of loops, i.e. paths that begin and end at a given reference point:

Definition 2.4. A *loop based at p* is path $\gamma: [0,1] \to X$ with $\gamma(0) = \gamma(1) = p$. The set of equivalence classes

$$\pi_1(X,p) := \{[\gamma] \mid \gamma \text{ is a loop in } X \text{ based at } p\}$$

is called the *fundamental group of X with respect to the base point p*.

Lemma 2.5. *The set $\pi_1(X,p)$ forms a group with respect to the composition of loops. Its neutral element is the class $[\varepsilon_p]$ of the constant loop $\varepsilon_p: [0,1] \to \{p\}$, and inverses are given by*

$$[\gamma]^{-1} = [\gamma^-] \quad \textit{for} \quad [\gamma] \in \pi_1(X,p).$$

Proof. Let $\varepsilon_p\colon[0,1]\to\{p\}$ be the constant loop. We claim that for all loops γ,μ,ν the following properties are satisfied:

- Associativity: $(\gamma\cdot\mu)\cdot\nu\sim\gamma\cdot(\mu\cdot\nu)$.
- Existence of a neutral element: $\varepsilon_p\cdot\gamma\sim\gamma\sim\gamma\cdot\varepsilon_p$.
- Existence of inverses: $\gamma\cdot\gamma^-\sim\varepsilon_p\sim\gamma^-\cdot\gamma$.

To see this one may use the homotopies that are given by the reparametrizations illustrated below:

[PICTURE OF THE ABOVE HOMOTOPIES]

□

Note that the passage to equivalence classes is essential for the statement in the above lemma: For a non-constant path $\gamma\colon[0,1]\to X$ the composite $\gamma^-\cdot\gamma$ is not constant, it is only *equivalent* to a constant path. Likewise the composition of paths is not associative, it is only so up to equivalence of paths. To simplify the notation for products of paths we will nonetheless omit brackets and make the convention that products are formed from left to right, i.e.

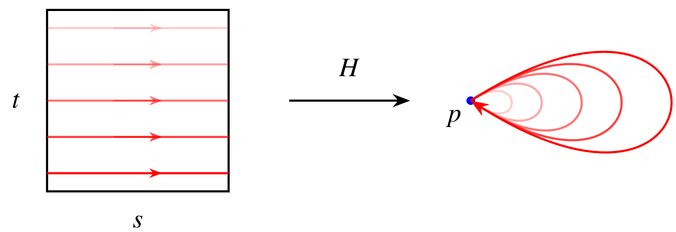
$$\gamma\cdot\mu\cdot\nu \;:=\; (\gamma\cdot\mu)\cdot\nu$$

etc. Of course any other way of putting brackets will lead to an equivalent path which is in fact a reparametrization of the previous one by a piecewise linear bijection from the unit interval onto itself.

Example 2.6. We have $\pi_1(\mathbb{R}^n,0)\simeq 0$ since any loop γ based at the origin in $\mathbb{R}^n$ is homotopic to the constant loop via the homotopy

$$H(s,t) \;:=\; (1-t)\cdot\gamma(s)$$

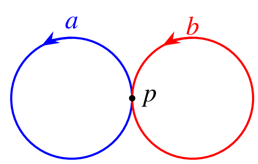
that contracts γ by rescaling it:



Example 2.7. We do not yet have tools to compute any non-trivial fundamental group, but let us already take an informal look at two intuitive examples:

a) We will prove soon that $\pi_1(S^1,1) \simeq \mathbb{Z}$ via the isomorphism that sends $n \in \mathbb{Z}$ to the loop
$$\gamma_n\colon \quad [0,1] \to S^1, \quad s \mapsto \exp(2\pi i n s).$$

b) In general the group $\pi_1(X,p)$ can be non-abelian. For instance, let $X = S^1 \vee S^1$ be the topological space obtained by glueing two circles together at a point p, then the two loops a and b around the two circles do not commute:

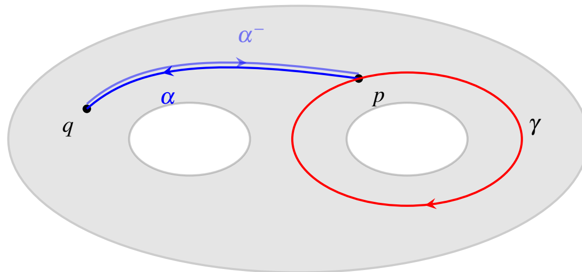


We will soon develop computational tools to prove this formally.

Before proceeding further towards the computation of fundamental groups, let us clarify how it depends on the choice of the base point. First, it is clear from the definitions that $\pi_1(X,p) = \pi_1(Y,p)$ where $Y \subset X$ denotes the path component of p in X, so we can only hope to compare fundamental groups for base points in the same path component. For those we have:

Lemma 2.8. *Let $p,q \in X$. Then any path $\alpha\colon [0,1] \to X$ from p to q gives rise to an isomorphism*
$$[-]^\alpha\colon \quad \pi_1(X,q) \xrightarrow{\sim} \pi_1(X,p), \quad [\gamma] \mapsto [\gamma]^\alpha := [\alpha^- \cdot \gamma \cdot \alpha]$$
of groups, and this isomorphism only depends on the equivalence class $[\alpha]$.

Proof. If $\gamma\colon [0,1] \to X$ is a loop based at p, then $\alpha^- \cdot \gamma \cdot \alpha$ is a loop based at $q = \alpha(1)$ as illustrated in the following picture:



By lemma 2.3 the equivalence class of this loop factors as

$$[\alpha^- \cdot \gamma \cdot \alpha] = [\alpha^-] \cdot [\gamma] \cdot [\alpha],$$

hence it only depends on the equivalence classes $[\alpha]$ and $[\gamma]$. Therefore we obtain a well-defined map

$$[-]^\alpha\colon \quad \pi_1(X,q) \to \pi_1(X,p), [\gamma] \mapsto [\alpha^- \cdot \gamma \cdot \alpha]$$

which depends only on $[\alpha]$, and an immediate computation shows that this map is a group homomorphism and has an inverse given by $[\gamma] \mapsto [\alpha \cdot \gamma \cdot \alpha^-]$. □

In particular, for a path-connected topological space X the isomorphism type of the group $\pi_1(X,p)$ does not depend on the choice of the point $p \in X$, so the following definition makes sense:

Definition 2.9. A topological space X is *simply connected* if it is path-connected and has trivial fundamental group.

This following alternative characterization of simply connected spaces explains their name. It also illustrates why in complex analysis simply connected subsets of the complex plane play an important role for the construction of primitives of holomorphic functions by path integration:

Lemma 2.10. *A topological space X is simply connected if and only if for any two points $a,b \in X$ there is a* unique *equivalence class of paths from a to b in X.*

Proof. If all points $a,b \in X$ can be connected by a path in X, then by definition X is pathwise-connected. If moreover this path is unique up to equivalence of paths, then taking $b = a$ we get that $\pi_1(X,a) \simeq \{0\}$, hence X is simply connected.

Conversely, if X is simply connected, then it is in particular path-connected, so for all $a,b \in X$ there exists a path in X from a to b. To show uniqueness of this connecting path up to equivalence, suppose we are given two paths $\gamma_0, \gamma_1 \colon [0,1] \to X$ from a to b. Then

$$\gamma = \gamma_1^- \cdot \gamma_0 \quad \text{is a loop based at } a.$$

By assumption $\pi_1(X,a) \simeq \{0\}$, hence it follows that γ is equivalent to a constant loop, which is equivalent to saying that $\gamma_1 \sim \gamma_0$ as desired. □

3 Functoriality and homotopy invariance

Let $f{:}X\to Y$ be a continuous map of topological spaces. Since the composite of continuous maps is again continuous, the composite of f with any path $\gamma{:}[0,1]\to X$ gives rise to a path

$$f\circ\gamma{:}\quad [0,1]\to Y,\quad t\mapsto f(\gamma(t)).$$

In the case of loops this gives a homomorphism between fundamental groups:

Lemma 3.1. *Let $p\in X$. Then any continuous map $f{:}X\to Y$ gives rise to a group homomorphism*

$$f_*{:}\quad \pi_1(X,p)\to\pi_1(Y,f(p)),\quad [\gamma]\mapsto[f\circ\gamma].$$

Proof. It follows from the definitions that for any two paths $\gamma_1,\gamma_2{:}[0,1]\to X$ we have the implication

$$\gamma_1\sim\gamma_2\quad\Longrightarrow\quad f\circ\gamma_1\sim f\circ\gamma_2.$$

Hence we get a well-defined map $f_*{:}\pi_1(X,p)\to\pi_1(Y,f(p)),[\gamma]\mapsto[f\circ\gamma]$. This is a group homomorphism:

$$\begin{aligned}((f\circ\gamma)\cdot(f\circ\mu))(s) &= \begin{cases}(f\circ\gamma)(2s) & \text{for } 0\le s<\frac{1}{2}\\ (f\circ\mu)(2s-1) & \text{for } \frac{1}{2}\le s\le 1\end{cases}\\ &= (f\circ(\gamma\cdot\mu))(s)\qquad \text{for all } s\in[0,1]\end{aligned}$$

implies $f_*(\alpha)\cdot f_*(\beta)=f_*(\alpha\cdot\beta)$ for $\alpha=[\gamma],\beta=[\mu]\in\pi_1(X,p)$. □

Since the fundamental group only sees loops up to continuous deformation, the above homomorphism f_* should remain unchanged if we deform the map $f{:}X\to Y$ continuously. This leads us to the following notion:

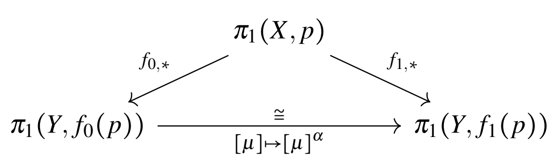
Definition 3.2. Let $f_0,f_1{:}X\to Y$ be two continuous maps. A *homotopy* between them is a continuous map

$$H{:}\ X\times[0,1]\to Y \ \text{ with } \ H(x,0)=f_0(x) \ \text{ and } \ H(x,1)=f_1(x) \ \text{ for all } x\in X.$$

We say that f_0 and f_1 are *homotopic* if there exists a homotopy between them.

Homotopic map induce 'the same' maps on fundamental groups, provided that we keep track of what happens to the respective base points:

Lemma 3.3. *Let $f_0,f_1{:}X\to Y$ be two continuous maps, and let $H{:}X\times[0,1]\to Y$ be a homotopy between them. Then for any point $p\in X$ we have a commutative diagram*



where the vertical arrow is induced by the path $\alpha\colon[0,1]\to Y, t\mapsto H(p,1-t)$.

Proof. Let $\gamma\colon[0,1]\to X$ be a loop based at the point p. We must show that the two loops

$$f_{1,*}(\gamma) \quad \text{and} \quad \alpha^{-}\cdot f_{0,*}(\gamma)\cdot\alpha$$

are equivalent. For this we must find a homotopy between them through loops based at $f_1(p)$. This can be done as follows:

[PICTURE OF THE HOMOTOPY DESCRIBED BELOW]

For $t\in[0,1]$, consider the path

$$\alpha_t\colon \quad [0,1] \to Y, \quad s \mapsto \alpha(st)$$

given by the initial segment of length t in the path α, reparametrized so that its domain is again the unit interval. This path begins at the point $\alpha_t(0)=f_1(p)$ and ends at $\alpha_t(1)=\alpha(t)$. On the other hand, for each t we can define a loop based at this end point via

$$\gamma_t\colon \quad [0,1] \to Y, \quad s \mapsto H(\gamma(s),1-t).$$

The loops $\delta_t := \alpha_t^{-}\cdot\gamma_t\cdot\alpha_t$ are then all based at the same point $f_1(p)$, and one checks that the map

$$[0,1]\times[0,1] \to Y \quad \text{given by} \quad (s,t) \mapsto \delta_t(s)$$

is a homotopy between the loops $\delta_0 = f_{1,*}(\gamma)$ and $\delta_1 = \alpha^{-}\cdot f_{0,*}(\gamma)\cdot\alpha$. □

In practice we often fix a point $p\in X$ and call the pair (X,p) a *pointed space*. By a map

$$f\colon \quad (X,p) \to (Y,q)$$

of pointed spaces we mean a continuous map $f\colon X\to Y$ with $f(p)=q$. For such maps the appropriate notion of deformation is homotopy through maps of pointed spaces. This is the special case $Z=\{p\}$ of the following notion:

Definition 3.4. Let $f_0, f_1\colon X\to Y$ be continuous maps, and let $H\colon X\times[0,1]\to Y$ be a homotopy between them. Suppose that $f_0|_Z = f_1|_Z$ for some subspace $Z\subset X$. We call H a *homotopy rel(ative to) Z* if

$$H(z,t) \quad \text{is independent of } t \text{ for each } z\in Z.$$

We then say that the two maps f_0 and f_1 are *homotopic rel(ative to) Z*.

Example 3.5. With the above notation,

a) two paths $\gamma_0, \gamma_1 \colon X = [0,1] \to Y$ are equivalent in the sense of definition 2.1 if and only if they are homotopic relative to $Z = \{0,1\} \subset X = [0,1]$.
b) two maps $f_0, f_1 \colon (X,p) \to (Y,q)$ of pointed spaces are homotopic through maps of pointed spaces if and only if they are homotopic relative to $Z = \{p\} \subset X$. In this case we also omit the set brackets and call the maps homotopic relative to p.

In the category of pointed spaces we therefore obtain:

Corollary 3.6. *If two maps of pointed spaces $f_0, f_1 \colon (X,p) \to (Y,q)$ are homotopic relative to the base point $Z = \{p\}$, then they induce on fundamental groups the same homomorphism*

$$f_{0,*} = f_{1,*} \colon \quad \pi_1(X,p) \to \pi_1(Y,q).$$

Proof. Saying that f_0 and f_1 are homotopic relative to p means by definition that we can choose a homotopy between them such that in corollary 3.3 the path α is constant. But then the map $\pi_1(Y,q) \to \pi_1(Y,q), [\mu] \mapsto [\mu]^\alpha$ is the identity. □

The above suggests that the notion of homeomorphism of topological spaces still keeps too much irrelevant information: What matters in topology is not whether a continuous map is invertible, but rather whether it is invertible up to homotopy. This leads to the following notion:

Definition 3.7. A continuous map $f \colon X \to Y$ is called a *homotopy equivalence* if there is a continuous map $g \colon Y \to X$ such that the composites $g \circ f$ and $f \circ g$ are homotopic to the identity maps id_X respectively id_Y. If such a homotopy equivalence exists, we also write

$$X \simeq Y$$

and say that the topological spaces X and Y are *homotopy equivalent*.

A topological space X is called *contractible* if it is homotopy equivalent to the space $Y = \{*\}$ consisting of a single point. The terminology is explained by the following observation:

Lemma 3.8. *A topological space X is contractible if and only if there exists $p \in X$ and a continuous map*

$$H \colon \quad X \times [0,1] \to X \quad \textit{with} \quad H(x,0) = p \quad \textit{and} \quad H(x,1) = x \quad \textit{for all} \quad x \in X.$$

In particular, every contractible space X is simply connected.

Proof. For a topological space $Y = \{*\}$ consisting of a single point,

- there is a unique map $f \colon X \to Y$, and
- giving a map $g \colon Y \to X$ amounts to giving a point $p = g(*) \in X$.

These maps are continuous for trivial reasons. Moreover we have $f \circ g = id_Y$ and the map $g \circ f : X \to \{p\} \subset X$ is constant. Therefore X is homotopy equivalent to a point if and only if the identity map id_X is homotopic to a constant map, which happens if and only if there exists a homotopy H with the stated properties. In this case every point $a \in X$ can be connected to the point p by the path

$$\gamma_a : [0,1] \to X, s \mapsto H(a,s),$$

hence all $a, b \in X$ can be connected by the path $\gamma_a^- \cdot \gamma_b$. So X is path-connected, and similarly one sees

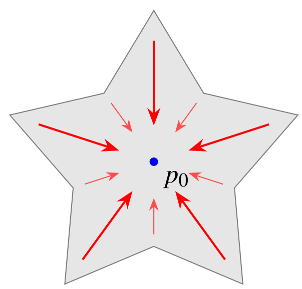
$$\pi_1(X,p) \simeq \{0\}$$

because any loop γ at p can be contracted via the homotopy $(s,t) \mapsto H(\gamma(s),t)$. □

Example 3.9. A subset $X \subset \mathbb{R}^n$ is called *star-shaped* if there is a point $p_0 \in X$ such that

$$\lambda p_0 + (1-\lambda)p \in X \quad \text{for every } p \in X \text{ and every } \lambda \in [0,1].$$

It follows from lemma 3.8 that any such subset is contractible:



Note that a star-shaped subset of $\mathbb{R}^n$ need not be convex: In fact, a subset $X \subset \mathbb{R}^n$ is convex if and only if it is star-shaped with respect to *every* point $p_0 \in X$.

Remark 3.10. As we noted at the end of lemma 3.8, any contractible space is simply connected. The converse is not true: For example, one easily checks that the sphere

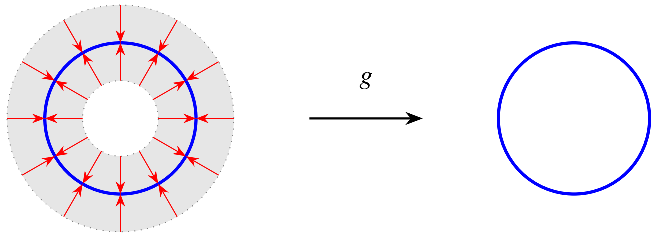
$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

is simply connected, but we will see later that it is not contractible.

Example 3.11. The unit circle is homotopy equivalent to an annulus, indeed the inclusion

$$f: \quad X = \{z \in \mathbb{C} \mid |z| = 1\} \hookrightarrow Y = \{z \in \mathbb{C} \mid \tfrac{1}{2} < |z| < \tfrac{3}{2}\}$$

is a homotopy equivalence — an inverse up to homotopy is $g : Y \to X, z \mapsto z/|z|$:



Note that by the same argument the unit circle is also homotopy equivalent to a Möbius strip. These two examples already give some idea about what information we lose in passing from homeomorphism to homotopy equivalence. However, all information about fundamental groups is kept:

Lemma 3.12. *If X and Y are homotopy equivalent topological spaces and both of them are path-connected, then their fundamental groups are isomorphic.*

Proof. By assumption there exist $f: X \to Y$ and $g: Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the identity map. Since the identity map of a topological space induces the identity map on fundamental groups, it then follows from lemma 3.3 that the two maps

$$(g \circ f)_*: \quad \pi_1(X,x_0) \to \pi_1(X,x_1) \quad \text{with} \quad x_1 = g(f(x_0))$$
$$(f \circ g)_*: \quad \pi_1(Y,y_0) \to \pi_1(Y,y_1) \quad \text{with} \quad y_1 = f(g(y_0))$$

are given by conjugation with certain paths α and β from x_0 to x_1 and from y_0 to y_1 respectively. In particular these two maps are isomorphisms of groups. This holds for any choice of base points $x_0 \in X$ and $y_0 \in Y$. Since for path-connected spaces the fundamental groups for different base points are isomorphic, we may as well assume that $y_0 = f(x_0)$. Then also $g(y_0) = g(f(x_0)) = x_1$ and hence $f(x_1) = f(g(y_0)) = y_1$, and the two above maps factor as

$$(g \circ f)_* = g_* \circ f_*: \quad \pi_1(X,x_0) \to \pi_1(Y,y_0) \to \pi_1(X,x_1),$$
$$(f \circ g)_* = f_* \circ g_*: \quad \pi_1(Y,y_0) \to \pi_1(X,x_1) \to \pi_1(Y,y_1).$$

The first row being an isomorphism implies that the map $g_*: \pi_1(Y,y_0) \to \pi_1(X,x_1)$ is surjective, while the second row being an isomorphism implies that the same map is injective. So $g_*: \pi_1(Y,y_0) \to \pi_1(X,x_1)$ is an isomorphism and we are done. □

4 The fundamental group of the circle

As a first nontrivial example we will now compute the fundamental group of the unit circle

$$S^1 \;=\; \{z \in \mathbb{C} \mid |z| = 1\}.$$

Our computation will be a special case of arguments that we will develop later in the chapter about covering spaces, but we here give an independent discussion of this special case. For each $n \in \mathbb{Z}$, the path

$$\gamma_n\colon \quad [0,1] \to S^1, \quad s \mapsto \exp(2\pi i n s)$$

is a loop based at the point 1, and we show:

Theorem 4.1. *We have an isomorphism* $\varphi\colon \mathbb{Z} \xrightarrow{\sim} \pi_1(S^1,1)$ *given by* $n \mapsto [\gamma_n]$.

Proof. One easily checks the identity $[\gamma_m]\cdot[\gamma_n] = [\gamma_{m+n}]$, hence the map φ is a group homomorphism. To show that it is in fact an isomorphism, consider the continuous map

$$p\colon \quad \mathbb{R} \to S^1, \quad s \mapsto \exp(2\pi i s).$$

It wraps each interval of length one once around the circle counterclockwise. For each $n \in \mathbb{Z}$ one has

$$\gamma_n \;=\; f \circ \tilde{\gamma}_n \quad \text{for the path} \quad \tilde{\gamma}_n\colon \quad [0,1] \to \mathbb{R}, \quad s \mapsto ns.$$

We also say that $\tilde{\gamma}_n$ is a path that *lifts* γ_n. Such lifts exist more generally:

> *For every path* $\gamma\colon [0,1] \to S^1$ *starting at the base point* $\gamma(0) = 1$*, there is a unique path*
>
> $$\tilde{\gamma}\colon \quad [0,1] \to X \quad \textit{with} \quad \tilde{\gamma}(0) \;=\; 0$$
>
> *such that* $\gamma = p \circ \tilde{\gamma}$ *as shown in the following commutative diagram:*

$$\begin{array}{ccc} & \overset{\exists!\tilde{\gamma}}{\nearrow} & \mathbb{R} \\ & & \downarrow p \\ [0,1] & \xrightarrow[\gamma]{} & S^1 \end{array} \qquad\qquad (*)$$

To see this, note that for any small enough path-connected open subset $U \subset S^1$ the preimage is a disjoint union

$$p^{-1}(U) \;=\; \bigsqcup_{n \in \mathbb{Z}} V_n$$

where each $V_n = V_0 + n \subset \mathbb{R}$ is an interval of length < 1 on which the map p restricts to a homeomorphism

$$p|_{V_n}\colon \quad V_n \xrightarrow{\sim} U.$$

So any path in $U \subset S^1$ lifts uniquely to a path in each of the intervals $V_n \subset \mathbb{R}$, and the choice of n is determined by a choice of the initial point of the lift. Let us call an open subset $U \subset S^1$ with the above properties *small*. Now by compactness $[0,1]$ can be covered by finitely many preimages $\gamma^{-1}(U)$ with $U \subset S^1$ small, hence γ is a composition of finitely many paths with image contained in small subsets. Each of those finitely many paths has a unique lift starting from a given lift of the starting point. The lifting property $(*)$ then follows by continuously gluing these unique lifts with given starting points for each of the finitely many appearing small subsets.

Having checked the lifting property $(*)$, let us now return to the proof of the theorem. To any loop $\gamma: [0,1] \to S^1$ based at 1 we may attach the end point $\tilde{\gamma}(1)$ of the unique lift $\tilde{\gamma}$ that we constructed above (starting at $\tilde{\gamma}(0) = 0$). Applying a lifting argument similar to the above, by suitably subdividing the compact set $[0,1] \times [0,1]$ which is the domain of a homotopy of loops we see that the value

$$\tilde{\gamma}(1) \in p^{-1}(1) = \mathbb{Z}$$

only depends on the equivalence class $[\gamma] \in \pi_1(S^1, 1)$. So we obtain a well-defined map

$$\deg: \quad S^1 \to \mathbb{Z}, \quad [\gamma] \mapsto \tilde{\gamma}(1)$$

which we call the *degree map*. One easily checks that the degree map is a group homomorphism. Moreover, by construction we have

$$\deg \circ \varphi = id_{\mathbb{Z}}: \quad \mathbb{Z} \xrightarrow{\varphi} \pi_1(S^1, 1) \xrightarrow{\deg} \mathbb{Z}.$$

Hence it suffices to show that deg is injective. To see this, suppose $\gamma_0, \gamma_1 : [0,1] \to S^1$ are two loops with $\deg(\gamma_0) = \deg(\gamma_1)$. Then the composite

$$\gamma := \gamma_1^- \cdot \gamma_0 \quad \text{has degree} \quad \deg(\gamma) = \deg(\gamma_0) - \deg(\gamma_1) = 0.$$

By definition this means that the unique lift $\tilde{\gamma}: [0,1] \to \mathbb{R}$ starting at the origin also ends at the origin, i.e. it is a loop in $\mathbb{R}$. Since the real line is contractible, it follows that $\tilde{\gamma}$ is homotopic to a constant loop. But then $\gamma = p \circ \tilde{\gamma}$ is also homotopic to a constant loop, which implies $[\gamma_1] = [\gamma_0]$ as desired. □

As a first application of the above, we can give a simple proof of the fact that the unit disk

$$D := \{z \in \mathbb{C} \mid |z| \le 1\}$$

cannot be retracted onto its outer boundary $\partial D = S^1$ without cutting a hole into it:

Corollary 4.2. *There is no continuous map $r: D \to \partial D$ with $r|_{\partial D} = id$.*

Proof. Let $i: \partial D \hookrightarrow D$ be the inclusion of the unit circle as the boundary of the unit disk. If there exists a continuous map $r: D \to \partial D$ with $r \circ i = id$, then by functoriality we have

$$r_* \circ i_* = id: \quad \pi_1(\partial D, 1) \xrightarrow{i_*} \pi_1(D, 1) \xrightarrow{r_*} \pi_1(\partial D, 1).$$

But $\pi_1(D,1) = \{0\}$ since D is contractible, while $\pi_1(\partial D, 1) = \pi_1(S^1, 1) \simeq \mathbb{Z} \neq \{0\}$ by the previous lemma. This gives a contradiction. □

Corollary 4.3 (Brouwer fixed point theorem). *Every continuous map $f: D \to D$ has at least one fixed point, i.e. there exists a point $p \in D$ with $f(p) = p$.*

Proof. Suppose $f: D \to D$ is a continuous map with no fixed point. Then $f(p) \neq p$ for any $p \in D$, so we can define a map $r: \quad D \to \partial D = S^1$ by sending a point p to be the unique point of intersection with S^1 of the half-ray from p to $f(p)$. Since f was assumed to be continuous, one easily checks that r is continuous in contradiction to the previous corollary. □

Another nice application of our computation of the fundamental group of the circle is a simple proof of the following result:

Corollary 4.4 (Fundamental theorem of algebra). *Let*

$$f(x) \;=\; x^n + c_{n-1}x^{n-1} + \cdots + c_0 \;\in\; \mathbb{C}[x]$$

be a monic polynomial of degree $n > 0$ with coefficients $c_0, \dots, c_{n-1} \in \mathbb{C}$. Then we have

$$f(a) \;=\; 0 \quad \textit{for some} \quad a \;\in\; \mathbb{C}.$$

Proof. Suppose that $f(a) \neq 0$ for all $a \in \mathbb{C}$. Using this for $a \in S^1$ we may define a continuous map

$$g\colon \quad S^1 \;\to\; S^1 \quad \text{by} \quad g(a) \;:=\; \frac{f(a)}{|f(a)|}.$$

We define the degree $\deg(g) \in \mathbb{Z}$ of this map to be the image of $[g \circ \exp] \in \pi_1(S^1, 1)$ under the isomorphism

$$\deg\colon \quad \pi_1(S^1, 1) \;\xrightarrow{\sim}\; \mathbb{Z}$$

from theorem 4.1. We will compute this degree in two different ways:

a) Since $f(a) \neq 0$ for $|a| < 1$, we may define a homotopy $h: S^1 \times [0,1] \to S^1$ by the formula

$$h(s,t) \;:=\; \frac{f_t(s)}{|f_t(s)|} \quad \text{for} \quad f_t(s) \;:=\; f(st).$$

This is a homotopy from g to a constant map, hence we get $\deg(g) = 0$.

b) Since $f(a) \neq 0$ for $|a| > 1$, we may define a homotopy $H: S^1 \times [0,1] \to S^1$ by the formula

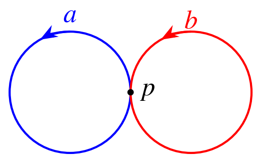
$$H(s,t) \;:=\; \frac{F_t(s)}{|F_t(s)|} \quad \text{for} \quad F_t(s) \;:=\; t^n f(s/t)$$
$$= s^n + t(c_1 s^{n-1} + t c_2 s^{n-2} + \cdots + t^{n-1} c_n).$$

This is a homotopy from g to the loop γ_n given by $s \mapsto s^n$, hence $\deg(g) = n$.

For $n > 0$ this gives a contradiction, so we must have $f(a) = 0$ for some $a \in \mathbb{C}$. □

5 The Seifert – van Kampen theorem

We now discuss an important tool to compute fundamental groups of topological spaces in terms of open covers. For instance, we will see that the fundamental group of the figure eight is a free group on the two generators a, b that are shown in the following picture:



In other words, any element of $\pi_1(X, p)$ can be written uniquely as a finite product

$$g_1 g_2 \cdots g_k \quad \text{with} \quad k \in \mathbb{N}_0 \quad \text{and} \quad g_1, \dots, g_k \in \{a, b\}.$$

This is an instance of the following abstract construction in group theory:

Definition 5.1. The *free product* $G_1 * G_2$ of two groups G_1 and G_2 is the set of all finite formal expressions

$$g_1 g_2 \cdots g_k \quad \text{with} \quad k \in \mathbb{N} \quad \text{and} \quad g_i \in G_1 \sqcup G_2,$$

modulo the equivalence relation $\sim$ that is generated by the following simplification rules for $\alpha = 1, 2$:

- Any occurence of the neutral element $g_i = e \in G_\alpha$ may be removed.
- For any two consecutive elements $g_i, g_{i+1} \in G_\alpha$ that both lie in the group G_α, the formal product of these two elements may be replaced by their actual product in that group, i.e.

$$\cdots g_i g_{i+1} \cdots \sim \cdots g \cdots \quad \text{for the element} \quad g = g_i g_{i+1} \in G_\alpha.$$

One immediately verifies that $G_1 * G_2$ is a group with respect to the product defined by

$$(g_1 g_2 \cdots g_k) \cdot (h_1 h_2 \cdots h_l) := g_1 g_2 \cdots g_k h_1 h_2 \cdots h_l.$$

Note that this group is non-abelian as soon as G_1 and G_2 are both non-trivial!

We have given an explicit construction, but the free product $G_1 * G_2$ can also be characterized more conceptually by the following universal property: For any other group G and any pair of group homomorphisms $\varphi_\alpha : G_\alpha \to G$, there is a unique group homomorphism

$$\varphi: \quad G_1 * G_2 \longrightarrow G$$

with $\varphi \circ \iota_\alpha = \varphi_\alpha$ for the two inclusions $\iota_\alpha : G_\alpha \hookrightarrow G_1 * G_2$. One easily sees that this property determines the free product uniquely up to canonical isomorphism. In practice we often want a variant where we only consider pairs of homomorphisms that agree on some given common subgroup of the two groups G_1 and G_2. This leads to the following notion:

Definition 5.2. Let H be a group. The *pushout* or *amalgamated product* of G_1, G_2 with respect to two group homomorphisms $f_\alpha : H \to G_\alpha$ ($\alpha = 1,2$) is defined to be the quotient

$$G_1 *_H G_2 := (G_1 * G_2)/N,$$

where we denote by

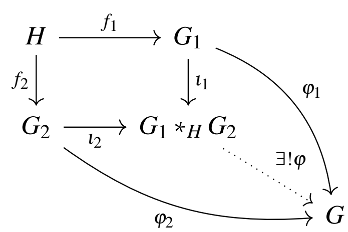
$$N \trianglelefteq G_1 * G_2$$

the smallest normal subgroup containing $f_1(h)^{-1} \cdot f_2(h)$ for all $h \in H$.

Lemma 5.3. *The pushout has the following universal property: For any group G and any pair of homomorphisms $\varphi_\alpha : G_\alpha \to G$ with $\varphi_1 \circ f_1 = \varphi_2 \circ f_2$, there is a unique homomorphism*

$$\varphi: \quad G_1 *_H G_2 \longrightarrow G$$

such that $\varphi \circ \iota_\alpha = \varphi_\alpha$ for the natural homomorphisms $\iota_\alpha : G_\alpha \to G_1 *_H G_2$:



Proof. Left as an exercise. □

Let us now come back to topology. The goal of this section is to compute the fundamental group of a topological space in terms of an open cover:

Theorem 5.4 (Seifert – van Kampen). *Consider a topological space with an open cover*

$$X = U_1 \cup U_2$$

where $U_1, U_2 \subset X$ are path-connected open subsets whose intersection $U_{12} = U_1 \cap U_2$ is also path-connected. Then for any $p \in U_{12}$ we have

$$\pi_1(X, p) \simeq \pi_1(U_1, p) *_{\pi_1(U_{12}, p)} \pi_1(U_2, p).$$

In fact, at the end of this section we will prove a generalized version of this theorem for covers involving more than two open subsets. But before coming to the proof, let us illustrate the theorem by some simple examples:

Example 5.5. For the above theorem the path-connectedness of $U_{12} = U_1 \cap U_2$ is crucial: For example, the unit circle $S^1 \subset \mathbb{C}$ can be covered by the simply connected open subsets

$$U_1 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > -\tfrac{1}{2}\} \quad \text{and} \quad U_2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) < \tfrac{1}{2}\}.$$

Here the groups $\pi_1(U_1, p)$ and $\pi_1(U_2, p)$ are trivial, hence so is their amalgamated product. But the Seifert – van Kampen theorem does not apply: Here $U_1 \cap U_2$ is not path-connected. Indeed

$$\pi_1(S^1, p) \simeq \mathbb{Z} \neq \{0\},$$

so the conclusion of the Seifert – van Kampen theorem fails in this case!

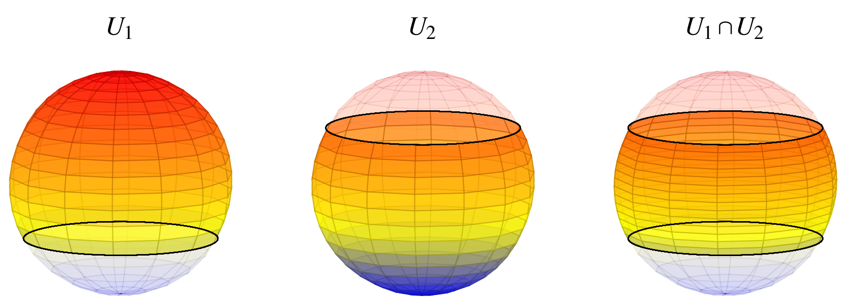
Example 5.6. For $n \geq 2$ the unit sphere $S^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$ is simply connected: It is obviously path-connected, hence we only need to verify that

$$\pi_1(S^n, p) \simeq \{0\} \quad \text{for all} \quad n > 1.$$

This follows easily by decomposing the sphere as a union $S^n = U_1 \cup U_2$ of the two open subsets

$$\begin{aligned} U_1 &= \{(x_1, \dots, x_n) \in S^n \mid x_n > -\tfrac{1}{2}\}, \\ U_2 &= \{(x_1, \dots, x_n) \in S^n \mid x_n < +\tfrac{1}{2}\}. \end{aligned}$$

as shown in the following picture:



Each of these subsets is homeomorphic to an open ball in $\mathbb{R}^{n-1}$ via a stereographic projection, so $\pi_1(U_1, p) \simeq \pi_1(U_2, p) \simeq \{0\}$. In contrast to the previous example, their intersection

$$U_{12} = U_1 \cap U_2 = \{(x_1, \dots, x_n) \in S^n \mid -\tfrac{1}{2} < x < \tfrac{1}{2}\}$$

is path-connected. Hence the Seifert – van Kampen theorem gives

$$\pi_1(S^n, p) \simeq \pi_1(U_1, p) *_{\pi_1(U_{12}, p)} \pi_1(U_2, p) \simeq \{0\}.$$

Example 5.7. If $X = U_1 \cup U_2$ with path-connected open subsets $U_1, U_2 \subset X$ whose intersection U_{12} is simply connected, then the Seifert – van Kampen theorem says

$$\pi_1(X, p) \simeq \pi_1(U_1, p) * \pi_1(U_2, p) \quad \text{for any} \quad p \in U_1 \cap U_2.$$

For instance, the fundamental group of the figure eight is the free group

$$\pi_1(\infty, p) \simeq \mathbb{Z} * \mathbb{Z}$$

on two generators as mentioned earlier. In general, define the *wedge sum* of two pointed topological spaces (X_α, p_α) to be the topological space

$$X_1 \vee X_2 := (X_1 \sqcup X_2)/\sim$$

where $\sim$ is the equivalence relation that identifies p_1 with p_2. If each $p_\alpha \in X_\alpha$ is a deformation retract of some open neighborhood

$$V_\alpha \subset X_\alpha$$

in the sense that the inclusion $\{p_\alpha\} \hookrightarrow V_\alpha$ is a homotopy equivalence, then the same argument as above shows that

$$\pi_1(X_1 \vee X_2, p) \simeq \pi_1(X_1, p_1) * \pi_1(X_2, p_2).$$

Inductively we then also obtain for any finite number of pointed spaces (X_α, p_α) whose base points are deformation retracts of some open neighborhoods, their wedge sum has fundamental group

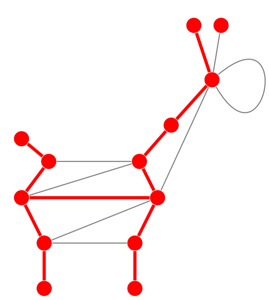
$$\pi_1(X_1 \vee \cdots \vee X_n, p) \simeq \pi_1(X_1, p_1) * \cdots * \pi_n(X_n, p_n),$$

where the iterated free product on the right hand side can be taken in any order since $*$ is associative. For instance, for a bouquet $X = S^1 \vee \cdots \vee S^1$ of n circles we get

$$\pi_1(S^1 \vee \cdots \vee S^1, p) \simeq \mathbb{Z} * \cdots * \mathbb{Z}.$$

This group is called the *free group on n generators*. It appears for instance as the fundamental group of finite connected graphs as in the following example:

Example 5.8. Let X be a *finite connected graph*, i.e. a connected topological space that is obtained from a finite set of vertices by attaching a finite number of edges such that each edge connects a vertex either to itself or to another vertex. By a *tree* in X we mean a subgraph $X_0 \subset X$ in which any two vertices are connected by a *unique* path, i.e. a subgraph which is simply connected. Any finite connected graph contains a maximal tree:



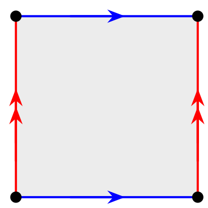
Moreover, any maximal tree $X_0 \subset X$ contains all the vertices of X. Fixing such a maximal tree, let n be the number of edges in $X \setminus X_0$. Since the endpoints of each edge lie on the chosen maximal tree, we obtain by contracting this tree to a point a map

$$f\colon \quad X \to S^1 \vee \cdots \vee S^1$$

to a bouquet of n circles. One may check that f is a homotopy equivalence, so by the previous example we get that

$$\pi_1(X,p) \simeq \mathbb{Z} * \cdots * \mathbb{Z}.$$

Example 5.9. A torus X can be obtained by taking a square and identifying pairs of opposite sides in a way preserving the orientation:



Consider the open cover $X = U_1 \cup U_2$ where $U_1 \subset X$ is a small open disk and $U_2 \subset X$ is the complement of a slightly smaller closed disk. Thus the subset U_1 is contractible while U_2 is homotopy equivalent to $S^1 \vee S^1$ as one may see by radial projection to the boundary of the unit square: Indeed, with the given identifications the boundary of the unit square becomes a bouquet of two circles. The intersection $U_{12} = U_1 \cap U_2$ is homotopy equivalent to a circle. If $\gamma\colon [0,1] \to U_{12}$ is a loop going round this circle once and $a, b \in \pi_1(U_2, *)$ are the two loops which are given by the two sides of the unit square, we may write the morphism induced by the inclusion $U_{12} \hookrightarrow U_2$ as

$$\pi_1(U_{12},*) = \langle \gamma \rangle \simeq \mathbb{Z} \to \pi_1(U_2,*) = \langle a,b \rangle \simeq \mathbb{Z} * \mathbb{Z}, \quad \gamma \mapsto aba^{-1}b^{-1}$$

So the Seifert – van Kampen theorem gives

$$\begin{aligned} \pi_1(X,*) &= \pi_1(U_1,*) *_{\pi_1(U_{12},*)} \pi_1(U_2,*) \\ &= \pi_1(U_2,*)/\pi_1(U_1 \cap U_2,*) = \langle a,b \mid aba^{-1}b^{-1} = 1 \rangle \simeq \mathbb{Z}^2. \end{aligned}$$

6 Proof of the Seifert – van Kampen theorem

The theorem of Seifert – van Kampen in the version that we stated above suffers from two drawbacks:

- the intersection $U_1 \cap U_2$ had to be path-connected, and
- we could only do covers by *two* open subsets at a time.

We now explain a generalization that deals with both issues in a more conceptual way, and then directly prove this generalized version.

The issue with path-connectedness is related to the fact that we want to fix the same base point for all fundamental groups: If $U_1 \cap U_2$ is not path-connected, then in which path component should we choose our base point? Conceptually it is better to use paths between arbitrary points. Such paths cannot always be composed, but there is a notion of a group-like object with a partially defined multiplication:

Definition 6.1. A *groupoid* G consists of the following data:

a) a non-empty set X of objects,

b) for all $a,b \in X$, a set $G_{a,b}$ of morphisms (for some a,b this set may be empty),

c) for all $a,b,c \in X$, composition maps

$$G_{a,b} \times G_{b,c} \to G_{a,c} \quad (\gamma,\mu) \mapsto \gamma \cdot \mu$$

satisfying associativity $(\gamma \cdot \mu) \cdot \lambda = \gamma \cdot (\mu \cdot \lambda)$ for any three composable γ,μ,λ,

d) for each $a \in X$, a neutral element $\varepsilon_a \in G_{a,a}$ such that

$$\varepsilon_a \cdot \gamma = \gamma \quad \text{and} \quad \mu \cdot \varepsilon_a = \mu \quad \text{for all } \gamma \in G_{a,b}, \mu \in G_{b,a},$$

e) for all $a,b \in X$, an inversion map $G_{a,b} \to G_{b,a}, \gamma \mapsto \gamma^{-1}$ such that

$$\gamma \cdot \gamma^{-1} = e_a \quad \text{and} \quad \gamma^{-1} \cdot \gamma = e_b \quad \text{for all } \gamma \in G_{a,b}.$$

In categorical language, items a) – d) amount to having a *small category*, and a groupoid is simply a small category in which every morphism is invertible. Any group can be viewed as a groupoid with only a single object. Groupoids with more objects arise naturally as follows:

Definition 6.2. Let X be a topological space. For a non-empty subset $X_0 \subset X$ we define the *fundamental groupoid*

$$G = \pi_1(X, X_0)$$

with base points in X_0 to be the following groupoid:

- Objects are the points in the subset $X_0 \subset X$.
- Morphisms between points $a,b \in X$ are given by

$$\pi_1(X,X_0)_{a,b} := \{\text{paths } \gamma\colon [0,1] \to X \text{ from } a \text{ to } b\}/\sim$$

  where $\sim$ is the equivalence relation of homotopy through paths from a to b.
- Neutral elements are given by constant paths, inverses by inverse paths.

When $X_0 = X$, we drop the base points from the notation and call $\Pi(X) = \pi_1(X,X)$ the *fundamental groupoid of* X. It will be useful in our discussion of covering spaces later. At the opposite extreme, we may also take $X_0 = \{p\}$ to be a singleton set. In this case we recover the notion of fundamentl groups:

Example 6.3. If $X_0 = \{p\}$ consists of a single point, then

$$\pi_1(X, \{p\}) = \pi_1(X, p)$$

when the group on the right hand side is viewed as a groupoid with a single object.

Using fundamental groupoids, we will get a version of the Seifert – van Kampen theorem without any path-connectedness assumptions. For this we have to view the fundamental groupoid as a functor from pairs of topological spaces to groupoids:

Remark 6.4. By a *pair of topological spaces* we mean pair (X, X_0), where X is a topological space and $X_0 \subset X$ is a subset endowed with the subspace topology. By defininition a *map of pairs*

$$f\colon \quad (V, V_0) \to (X, X_0)$$

is a continuous map $f\colon V \to X$ with $f(V_0) \subset X_0$. Any such map of pairs induces a morphism of groupoids

$$f_*\colon \quad \pi_1(V, V_0) \to \pi_1(X, X_0),$$

where the notion of morphisms of groupoids is defined in the obvious way.

In particular, given a pair of topological spaces (X, X_0) and a cover $X = U_1 \cup U_2$ by open subsets $U_1, U_2 \subset X$, we can apply the above remark to all the maps of pairs in the diagram

$$\begin{array}{ccc} (U_{12}, U_{12} \cap X_0) & \longrightarrow & (U_2, U_2 \cap X_0) \\ \downarrow & & \downarrow \\ (U_1, U_1 \cap X_0) & \longrightarrow & (X, X_0) \end{array}$$

with $U_{12} = U_1 \cap U_2$ to get a corresponding diagram of groupoids:

$$\begin{array}{ccc} \pi_1(U_{12}, U_{12} \cap X_0) & \longrightarrow & \pi_1(U_2, U_2 \cap X_0) \\ \downarrow & & \downarrow \\ \pi_1(U_1, U_1 \cap X_0) & \longrightarrow & \pi_1(X, X_0) \end{array}$$

The condition $X = U_1 \cup U_2$ amounts to saying that the first diagram is a pushout in the category of pairs of topological spaces, in the sense that the universal property analogous to the one in lemma 5.3 holds. The essence of the Seifert – van Kampen theorem is that under suitable assumptions the second diagram is a pushout in the category of groupoids, or as a slogan: The fundamental groupoid commutes with pushouts. This formulation of the theorem immediately generalizes to arbitrary open covers

$$X = \bigcup_{i \in I} U_i$$

if we replace the pushout by the notion of a coequalizer:

Definition 6.5. Let $\mathcal{C}$ be a category. Suppose we are given an index set I together with

- an object $U_i \in \mathcal{C}$ for every $\alpha \in I$,
- an object $U_{ij} = U_{ji} \in \mathcal{C}$ with a morphism $f_{ij}: U_{ij} \to U_i$ for all $i, j \in I$

Then an object $X \in \mathcal{C}$ with morphisms $f_i: U_i \to X$ is a *coequalizer* of the f_{ij} if it has the following universal property:

- We have $f_i \circ f_{ij} = f_i \circ f_{ji}$ for all $i, j \in I$.
- For any other $Z \in \mathcal{C}$ and $g_i: U_i \to Z$ such that $g_i \circ f_{ij} = g_i \circ f_{ji}$ for all $i, j \in I$, there is a unique morphism

$$g: \quad X \to Z \quad \text{with} \quad g_i = g \circ f_i \quad \text{for all } i \in I.$$

If such a coequalizer exists, it is determined uniquely up to unique isomorphism and we write

$$X = \operatorname{colim}_{i,j \in I} \Big(U_{ij} \rightrightarrows U_i \Big).$$

For $I = \{1, 2\}$ we also call say that X is the *pushout* of the diagram $U_1 \leftarrow U_{12} \to U_2$.

For instance, taking $\mathcal{C}$ to be the category of topological spaces, we can view any topological space as the colimit of an arbitrary open cover. The general form of the Seifert – van Kampen theorem says that in this case the fundamental groupoid of the space is the colimit of the fundamental groupoids of the covering sets:

Theorem 6.6. *Let (X, X_0) be a pair of topological spaces, and let $X = \bigcup_{i \in I} U_i$ be a cover by open subsets such that the subset X_0 meets every path component of the intersections $U_{ijk} = U_i \cap U_j \cap U_k$ for all triples of indices $i, j, k \in I$. Then*

$$\pi_1(X, X_0) = \operatorname{colim}_{i,j \in J} \Big(\pi_1(U_{ij}, U_{ij} \cap X_0) \rightrightarrows \pi_1(U_i, U_i \cap X_0) \Big).$$

Proof. We have to verify that $\pi_1(X, X_0)$ has the universal property required for the coequalizer. So let G be any groupoid with morphisms

$$g_i: \quad \pi_1(U_i, U_i \cap X_0) \to G$$

of groupoids such that for all $i, j \in I$ the following diagrams commute:

$$\begin{array}{ccc} \pi_1(U_{ij}, U_{ij} \cap X_0) & \xrightarrow{f_{ji}} & \pi_1(U_j, U_j \cap X_0) \\ {\scriptstyle f_{ij}} \downarrow & & \downarrow {\scriptstyle g_j} \\ \pi_1(U_i, U_i \cap X_0) & \xrightarrow[g_i]{} & G \end{array}$$

Here f_{ij} denotes the homomorphism induced by the inclusion $U_{ij} \hookrightarrow U_i$. We have to show that there is a unique morphism

$$g: \quad \pi_1(X, X_0) \to G$$

of groupoids with the property that

$$g_i = g \circ f_i: \quad \pi_1(U_i, U_i \cap X_0) \to \pi_1(X, X_0) \to G \quad \text{for all } i \in I,$$

where f_i denotes the homomorphism induced by the inclusion $U_i \hookrightarrow X$.

We begin by proving the uniqueness: Let $[\gamma] \in \pi_1(X, X_0)(a, b)$ be the homotopy class of a path

$$\gamma: \quad [0,1] \to X \quad \text{between two points } a, b \in X_0.$$

The preimages $\gamma^{-1}(U_i)$ with $i \in I$ form an open cover of the unit interval, hence by compactness of that interval we may find a subdivision $0 = s_0 < s_1 < \cdots < s_m = 1$ such that for each $\alpha \in \{0, 1, \dots, m-1\}$ the given path restricts on the corresponding subinterval to a path

$$\gamma_\alpha: \quad [s_\alpha, s_{\alpha+1}] \to U_{i_\alpha} \subset X \quad \text{for some} \quad i_\alpha \in I.$$

To simplify the notation, we fix such an index i_α for each α and put $V_\alpha = U_{i_\alpha}$ in what follows. Note that the end points of γ_α do not necessarily lie in X_0. But by assumption X_0 meets each path component of the intersections

$$V_\alpha \cap V_{\alpha+1} \subset X,$$

so we can find paths

$$\mu_\alpha: [0,1] \to V_\alpha \cap V_{\alpha+1} \quad \text{with} \quad \mu_\alpha(0) = \gamma(s_\alpha) \quad \text{and} \quad \mu_\alpha(1) \in X_0.$$

Moreover, we can assume μ_0 and μ_m are the constant paths at a resp. b. Then

$$\gamma \sim \gamma_0 \cdot \gamma_1 \cdots \gamma_{m-1} \sim \beta_0 \cdot \beta_1 \cdots \beta_{m-1} \quad \text{for the paths} \quad \beta_\alpha := \mu_\alpha^- \cdot \gamma_\alpha \cdot \mu_{\alpha+1}.$$

Now each β_α is a path inside $V_\alpha = U_{i_\alpha}$ and its two endpoints lie in X_0. Hence any morphism

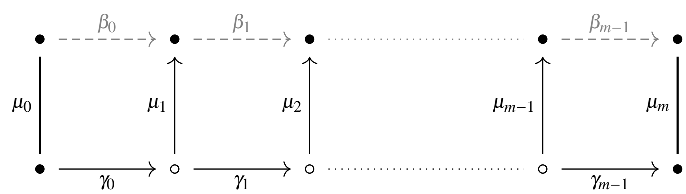
$$g: \quad \pi_1(X, X_0) \to G \quad \text{inducing the given} \quad g_{i_\alpha} = g \circ f_{i_\alpha}$$

which induces the given morphisms $g_{i_\alpha} = g \circ f_{i_\alpha}$ must satisfy

$$g[\gamma] = g_{i_0}[\beta_0] \cdot g_{i_1}[\beta_1] \cdots g_{i_{m-1}}[\beta_{m-1}] \quad \text{for the images} \quad g_{i_\alpha}[\beta_\alpha] \in G.$$

Since γ was an arbitrary path, this shows the uniqueness of the morphism g.

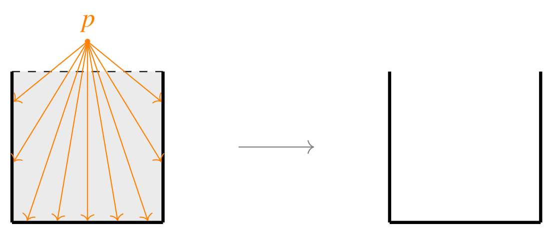
To show existence, we have to check that the expression that we obtained for $g[\gamma]$ only depends on the homotopy class $[\gamma]$ but not on the choice of the factorization of the path. As a preparation, let us draw our homotopy $\gamma \sim \beta_0\beta_1\cdots\beta_{m-1}$ as follows:



Here $\bullet$ indicates points whose image lies in X_0, while $\circ$ denotes arbitrary points. The homotopy is given on the solid boundary part

$$\Delta_\alpha := [0,1]\times\{0\}\cup\{0,1\}\times[0,1] \subset Q_\alpha = [0,1]\times[0,1]$$

of each of the small squares by the paths μ_α, γ_α, $\mu_{\alpha+1}$ as shown (with the thick lines on the very left and on the very right denoting the constant paths μ_0, μ_m), and we extend the homotopy continuously from this boundary part to the full square by precomposing with the retraction $r_\alpha\colon \quad Q_\alpha \to \Delta_\alpha$ that is given by the projection from some point above the square:



Such filling arguments will be used repeatedly in what follows. Returning to the independence of choices, suppose we are given paths $\gamma,\gamma'\colon[0,1]\to X$ from a to b in the same homotopy class

$$[\gamma] = [\gamma'] \in \pi_1(X,X_0)(a,b)$$

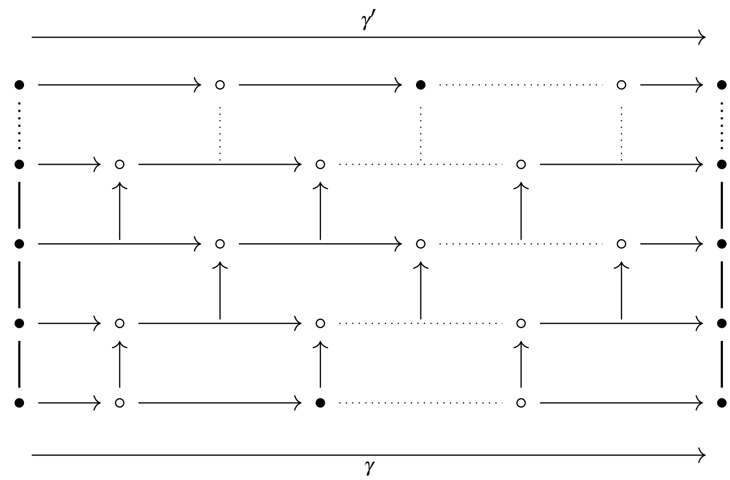
and two decompositions

$$\gamma = \beta_0\beta_1\cdots\beta_m \quad \text{and} \quad \gamma' = \beta_0'\beta_1'\cdots\beta_n'$$

where each $\beta_\alpha,\beta_\alpha'$ lies in one of the open subsets U_i and has end points in X_0. Pick a homotopy

$$H\colon \quad Q = [0,1]\times[0,1] \to X \quad \text{from} \quad \gamma \quad \text{to} \quad \gamma'.$$

By compactness we may subdivide the unit square into smaller rectangles such that each of the smaller rectangles is mapped into one of the open subsets U_i. We may assume that our chosen subdivision refines the given subdivision into a product of paths on $[0,1]\times\{0,1\}$. By slightly perturbing all except for the top and bottom rows to obtain a 'brick wall decomposition' as shown below, we may assume any point of the domain of our homotopy lies on at most *three* of the rectangles:



Here the left and the right vertical boundary are constant paths. The bottom and top row are given by γ and γ' respectively. On those top and bottom rows, all nodes corresponding to end points of $\beta_0,\dots,\beta_m$ and $\beta_0',\dots,\beta_n'$ are labelled $\bullet$, but all other nodes on those rows and in the interior of the square can only be labelled $\circ$.

The idea is now to deform the homotopy H to a new homotopy via a continuous map

$$\tilde{H}\colon \quad Q\times[0,1] \to X \quad \text{with} \quad \tilde{H}|_{Q\times\{0\}} = H$$

such that the original subdivision of the square is kept but all the labels $\circ$ can be replaced by $\bullet$. More precisely, we impose the following properties:

- If $R\subset Q$ is a rectangle in our subdivision with $H(R)\subset U_i$ for some $i=i_\alpha\in I$, then

$$\tilde{H}(R,t) \subset U_i \quad \text{for all} \quad t\in[0,1].$$

- For any node $p\in Q$ of our subdivision we have $\tilde{H}(p,1)\in X_0$.
- If the node already satisfied $H(p)\in X_0$, then $\tilde{H}(p,t)=H(p)$ for all t.

Such a homotopy $\tilde{H}$ can be constructed as follows: For each rectangle $R\subset Q$ in our subdivision of the square, fix an index $i=i(R)\in I$ with $H(R)\subset U_i$. Given any node p in our subdivision, we have ensured by the brick wall decomposition that the node lies on at most three different rectangles R_1,R_2,R_3. Put $i_\nu=i(R_\nu)\in I$. By our assumption X_0 meets each path component of

$$U = U_{i_1} \cap U_{i_2} \cap U_{i_3},$$

so we may choose a path $\mu_p\colon[0,1] \to U$ joining $H(p)$ to a point in X_0. If $H(p) \in X_0$, we take this path to be constant. Using these paths, we can now define the desired homotopy by proceeding on each brick $R \times [0,1]$ separately:

- On the bottom face $R \times \{0\}$ it is given by $\tilde{H}(x,0) = H(x)$.
- For each vertex p of the bottom face, put the path μ_p on the corresponding vertical edge of the brick, and extend this to a continuous map on each of the lateral faces using the retraction argment from before.
- Finally, fill in the entire brick via the retraction given again by projection from a point slightly above the brick, as we did previously for the retraction from a square but now in three dimensions.

If we apply this construction in each of the rectangles of our subdivision, then on the top face $Q \times \{1\}$ of the cube $Q \times [0,1]$ we obtain a homotopy

$$\tilde{Q}|_{Q\times\{1\}}\colon \quad Q \to X$$

which is a homotopy between two paths in the same classes as γ and γ' relative to the end points, and with the property that this homotopy sends all the nodes in our given subdivision to points of A_0. This new homotopy therefore sends each rectangle $R \subset Q$ in our subdivision to a commutative square in the groupoid $\pi_1(U_i, U_i \cap X_0)$ for $i = i(R) \in I$. Its image under the given morphism $g_i\colon \pi_1(U_i, U_i \cap X_0) \to G$ is then a commutative square in G. By combining all these commutative squares in the groupoid G we get

$$g_{i_0}[\beta_0]\cdots g_{i_m}[\beta_m] = g_{i'_0}[\beta'_0]\cdots g_{i'_n}[\beta' n'],$$

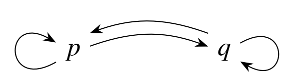
which proves that $g\colon \pi_1(X, X_0) \to G$ is well-defined. □

Example 6.7. From the above version of the Seifert – van Kampen theorem, we may give another computation of the fundamental group of the circle using the open cover $S^1 = U_1 \cup U_2$ with

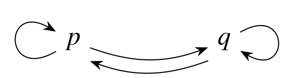
$$U_1 = \{(x,y) \in S^1 \mid y > -1/2\} \quad \text{and} \quad U_2 = \{(x,y) \in S^1 \mid y < 1/2\}.$$

As a set of base points, take $X_0 = \{p,q\}$ with $p = (1,0)$ and $q = (-1,0)$. In this case the fundamental groupoids of the open subsets have the following shape where the nodes p,q stand for objects (= points) and the arrows for morphisms (= paths):

$\pi_1(U_1, \{p,q\})$:



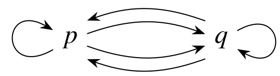
$\pi_1(U_2, \{p,q\})$:



$\pi_1(U_1 \cap U_2, \{p, q\})$:

The coequalizer is then the following groupoid:

$\pi_1(S^1, \{p, q\})$:



If we restrict to morphisms starting and ending at p, we see that $\pi_1(S^1, p) \simeq \mathbb{Z}$ is generated by a single loop which is the product ab in the above picture.

For reference, let us also state the fundamental group version of the Seifert – van Kampen theorem in the case of covers by more than two open sets. Given a family of groups $(G_i)_{i \in I}$ indexed by an arbitrary set I, we define their *free product*

$$\mathop{\ast}_{i \in I} G_i$$

to be the set of all formal expressions

$$g_1 g_2 \cdots g_k \quad \text{with} \quad k \in \mathbb{N} \quad \text{and} \quad g_1, \ldots, g_k \in \bigsqcup_{i \in I} G_i,$$

modulo the same relations as in definition 5.1. With this notation we have:

Corollary 6.8 (Seifert – van Kampen). *Let X be a topological space that admits a cover*

$$X = \bigcup_{i \in I} U_i$$

by open subsets $U_i \subset X$ such that for all $i, j, k \in I$, the intersections $U_i \cap U_j \cap U_k$ are path-connected. Suppose that there is a point $p \in \bigcap_{i \in I} U_i$, and consider the group homomorphism

$$\Phi\colon \quad G = \mathop{\ast}_{i \in I} \pi_1(U_i, p) \longrightarrow \pi_1(X, p)$$

induced by the inclusions $U_i \hookrightarrow X$. Then Φ induces an isomorphism

$$\overline{\Phi}\colon \quad G/N \xrightarrow{\sim} \pi_1(X, p)$$

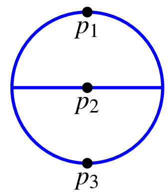
where $N \trianglelefteq G$ is the smallest normal subgroup containing for all $\gamma \in \pi_1(U_i \cap U_j, p)$ the elements

$$\iota_{ij*}(\gamma)^{-1} \cdot \iota_{ji*}(\gamma) \quad \textit{for the inclusions} \quad \iota_{ij}\colon U_i \cap U_j \hookrightarrow U_i.$$

The example of $X = S^1$ has already shown that in general the path-connectedness of the double intersections $U_i \cap U_j$ is needed for the above statement. To see that the path-connectedness of the triple intersections $U_i \cap U_j \cap U_k$ is also essential, consider the topological space

$$X = S^1 \cup [-1, 1] \subset \mathbb{R}^2$$

shown below:



For $i \in \{1,2,3\}$, let $U_i = X \smallsetminus \{p_i\}$ be the complement of the point p_i shown in the picture. Then we have

$$X = U_{-1} \cup U_0 \cup U_1 \quad \text{but} \quad \pi_1(X,*) \simeq \mathbb{Z} * \mathbb{Z} \not\simeq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}.$$

Chapter III
Covering spaces

1 Covering spaces and lifting

When we computed the fundamental group of the circle in theorem 4.1, the key point was to consider lifts of paths under the map

$$p\colon \quad \mathbb{R} \to S^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \quad x \mapsto \exp(2\pi i x).$$

The basic feature which guaranteed the existence and uniqueness of lifts was the observation that this map is a covering map in the following sense:

Definition 1.1. A continuous map $p\colon Y \to X$ between topological spaces is called a *covering map* or a topological *cover* if it is surjective and every $x \in X$ has an open neighborhood $U \subset X$ such that $p^{-1}(U) \subset Y$ is a disjoint union of open subsets each of which is mapped by p homeomorphically onto the open subset $U \subset X$. We call any path-connected open $U \subset B$ with this property a *fundamental neighborhood* and call $Y_x = p^{-1}(x)$ the *fiber* over the point $x \in X$.

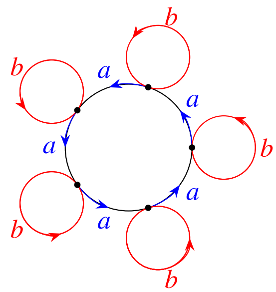
For any set F endowed with the discrete topology, the projection $p\colon X \times F \to X$ is a covering map. We also refer to this as a *trivial cover*. Any covering map can be identified locally on the target with a trivial cover, but globally there are usually many nontrivial covers:

Example 1.2. For any $n \in \mathbb{N}$ the map $p_n\colon S^1 \to S^1, z \mapsto z^n$ is a covering map. We have a commutative diagram

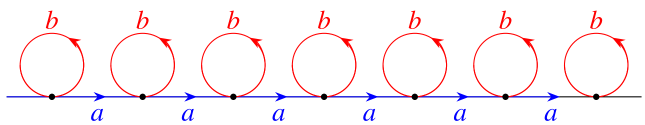
$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\exists q_n} & S^1 \\ & \searrow^{p} & \downarrow{\scriptstyle p_n} \\ & & S^1 \end{array}$$

where $q_n\colon \mathbb{R} \to S^1$ denotes the unique covering map with $q_n(0) = 1$, so all these covers can be understood in terms of the exponential cover considered above.

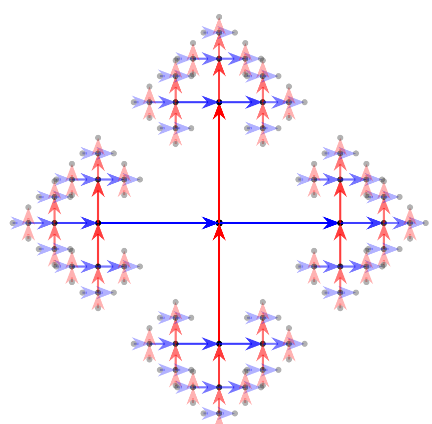
Example 1.3. The figure eight graph $X = S^1 \wedge S^1$ has many interesting covers. For instance, we can put the n-fold cover from the previous example over one of the two circles and attach a copy of the other circle at each of the n nodes, as illustrated in the following picture for $n = 5$:



We could also take the universal cover of one circle and attach a copy of the other circle over each of the countably many nodes:



We could also mix these constructions: For instance, we could replace each loop b in the previous picture by an n-fold cover and accordingly attach $n-1$ blue circles at each of those loops, etc. The 'largest' path-connected cover of the figure eight is obtained by gluing together infinitely many copies of the universal cover of each of the two unit circles. The result will be the infinite tree sketched below, where we omitted the labels a, b and rescaled the segments at each iteration:



Again each of the previous covers of $X = S^1 \vee S^1$ can be obtained from this last cover by suitable identifications between the infinitely many edges and vertices.

The crucial ingredient in our computation of $\pi_1(S^1,*)$ in the proof of theorem 4.1 was that every path in the unit circle admits a unique lift under the exponential map, once a lift of the starting point is fixed. The same holds for arbitrary covers:

Theorem 1.4 (Path Lifting Property). *Let $p: Y \to X$ be a covering map. Fix $x \in X$ and a point*

$$y \in p^{-1}(x).$$

in the fiber above it. Then the following properties hold:

a) For any path $\gamma: [0,1] \to X$ with $\gamma(0) = x$, there is a unique path $\tilde{\gamma}: [0,1] \to Y$ with

$$p \circ \tilde{\gamma} = \gamma \quad \text{and} \quad \tilde{\gamma}(0) = y.$$

b) Every homotopy $\gamma \sim \mu$ lifts under $p: Y \to X$ to a unique homotopy $\tilde{\gamma} \sim \tilde{\mu}$.

Proof. For part $a)$, note that by compactness of the unit interval we may find for any path γ a subdivision $0 = s_0 < s_1 < \cdots < s_m = 1$ of the unit interval such that γ maps each interval $[s_{i-1}, s_i]$ into some fundamental neighborhood $U_i \subset X$. Thus for each i the preimage $p^{-1}(U_i) = \sqcup_j V_{ij}$ is a disjoint union of open subsets $V_{ij} \subset Y$ such that each

$$p_{ij} \;=\; p|_{V_{ij}}: \quad V_{ij} \;\longrightarrow\; U_i$$

is a homeomorphism. We can then lift the path inductively as follows:

- Put $\tilde{\gamma}(0) = y$.
- Inductively, if $\tilde{\gamma}$ has already been defined on the interval $[s_0, s_i]$ for some i, let j be the unique index with $\tilde{\gamma}(s_i) \in V_{ij}$, and define

$$\tilde{\gamma}|_{[s_i, s_{i+1}]} \;:=\; p_{ij}^{-1} \circ \gamma|_{[s_i, s_{i+1}]}: \quad [s_i, s_{i+1}] \to V_{ij}.$$

Part $b)$ follows in the same way: Given a homotopy $H: Q = [0,1] \times [0,1] \to X$, we subdivide Q into smaller squares that map into fundamental open neighborhoods and then lift the homotopy on each of those smaller squares as above. Note that both in $a)$ and $b)$ the uniqueness of the lift is also clear by the same argument. □

We will classify all covers of a given topological space in terms of subgroups of its fundamental group, starting from the following observation:

Corollary 1.5. *Let $p: Y \to X$ be a cover. Then for any $x \in X$ and $y \in f^{-1}(x)$, the induced map*

$$p_*: \pi_1(Y, y) \to \pi_1(X, x) \quad \textit{is injective}.$$

Proof. If $\tilde{\gamma}: [0,1] \to Y$ is a loop based at y whose image $\gamma = p \circ \tilde{\gamma}$ is homotopic to a constant loop through a homotopy keeping the base point fixed, then by the previous lemma the homotopy can be lifted to a homotopy from $\tilde{\gamma}$ to the constant loop at $y = \tilde{\gamma}(0)$, and the lifted homotopy again keeps the base points fixed. □

Example 1.6. The covers $p: \mathbb{R} \to S^1$ and $p_n: S^1 \to S^1$ of the unit circle give rise to the subgroups

$$\mathrm{im}(p_*) \;=\; \{0\} \;\subset\; \mathrm{im}(p_{n,*}) \;=\; n\mathbb{Z} \;\subset\; \pi_1(S^1, 1) \;=\; \mathbb{Z}.$$

Example 1.7. The covers of $X = S^1 \vee S^1$ that we considered in example 1.3 give rise to the subgroups

$$\begin{aligned}\{1\} &\subset \{a^{e_1}b^{f_1}\cdots a^{e_k}b^{f_k} \mid e_1+\cdots+e_k = 0\}\\ &\subset \{a^{e_1}b^{f_1}\cdots a^{e_k}b^{f_k} \mid e_1+\cdots+e_k \equiv 0 \mod n\}\\ &\subset \langle a,b\rangle \;=\; \pi_1(X,*).\end{aligned}$$

Note that unlike in the previous example, there are many more subgroups of the free group $\langle a,b\rangle = \mathbb{Z} * \mathbb{Z}$, and indeed the figure eight graph $X = S^1 \vee S^1$ has many more covers than the examples considered above!

2 Abstract covering theory via groupoids

The correspondence between covers of a topological space and subgroups of its fundamental group can be phrased in the language of groupoids, which separates the formal algebraic aspects from the geometry. The key property of coverings is the lifting property for paths with given starting point but variable end point. These paths form a star in the fundamental groupoid:

Definition 2.1. Let G be a groupoid and X its set of objects. For any $a \in X$, we define its *star* to be the set

$$G_{a,\star} \;:=\; \{(b,\gamma) \mid b \in X,\ \gamma \in G_{a,b}\}$$

of all morphisms $\gamma\colon a \to b$ from the given point a to arbitrary other points $b \in X$.

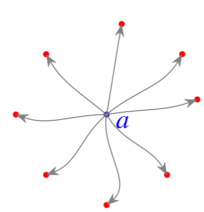
Example 2.2. For a topological space X, let $\Pi(X) = \Pi(X,X)$ be its fundamental groupoid of homotopy classes of paths with arbitrary starting and end points. The star of $a \in X$ is

$$\Pi(X)_{a,\star} \;=\; \big\{\text{ paths } \gamma\colon [0,1] \to X \mid \gamma(0) = a \,\big\}/\sim$$

where $\sim$ denotes homotopy rel $\{0,1\}$. If X is simply connected, then the evaluation map

$$\Pi(X)_{a,\star} \;\to\; X, \quad [\gamma] \;\mapsto\; \gamma(1)$$

is a bijection. Fixing a path in each homotopy class, we get the following picture of the star:



Note that if X is not simply connected, then the evaluation map will not be injective.

In the abstract framework of groupoids, the notion of a covering map has the following counterpart which captures the path lifting property:

Definition 2.3. A *cover* of a groupoid G is a morphism $p{:}H \to G$ from another groupoid such that

- p induces a surjection between the underlying sets of objects, and
- p induces a bijection of stars

$$p|_{H_{y,\star}}: \quad H_{y,\star} \to G_{x,\star} \quad \text{for each object } y \text{ of } H \text{ with image } x = p(y).$$

Note that in this case, in terms of the fiber $p^{-1}(x) = \{\text{objects } y \text{ of } H \mid p(y) = x\}$ we have

$$p^{-1}(G_{x,\star}) = \bigsqcup_{y \in p^{-1}(x)} H_{y,\star}.$$

We are interested in the following example:

Lemma 2.4. *For any covering map $p{:}Y \to X$ of topological spaces, the induced morphism of fundamental groupoids*

$$\Pi(Y) \to \Pi(X) \quad \textit{is a covering of groupoids.}$$

Proof. Immediate by the unique path lifting property in theorem 1.4. □

In the previous section we have seen that any cover of topological spaces induces an injective map between fundamental groups. The same works more generally for covers of groupoids. More precisely, for an object x in a groupoid G denote its automorphism group by

$$G_a := G_{a,a}.$$

If $G = \Pi(X)$ is the fundamental groupoid of a topological space, we recover the fundamental group

$$G_a = \pi_1(X, a)$$

with base point a. In general we have:

Proposition 2.5. *Let $p{:}H \to G$ be a cover of groupoids and x an object of G.*

a) For any $y \in p^{-1}(x)$, the induced morphism $p_*{:}H_y \to G_x$ is injective.

b) For any $z \in p^{-1}(x)$ with $H_{y,z} \neq \varnothing$, the subgroup

$$p_*(H_z) \le G_x \quad \textit{is conjugate to} \quad p_*(H_y) \le G_x.$$

c) Conversely, all conjugates of the subgroup $p_*(H_y) \le G_x$ arise like this.

Proof. Part $a)$ follows immediately from the injectivity of the map $H_{y,\star} \to G_{x,\star}$ between stars. For part $b)$ pick any $\alpha \in H_{y,z}$. Then conjugation by α gives a group isomorphism

$$H_y \to H_z, \quad \gamma \mapsto \alpha^{-1}\gamma\alpha$$

that maps under p to the conjugation by the image $\beta \in G_{x,x} = G_x$ of α. For part $c)$ note that by the surjectivity of the map $H_{y,\star} \to G_{x,\star}$ between stars, any $\beta \in G_x = G_{x,x}$ is the image of some $\alpha \in H_{y,\star}$, hence the computation from part $b)$ applies. □

Since any topological space can be decomposed as a disjoint union of its path components, the discussion of covering maps easily reduces to the case where all spaces in question are path-connected. The notion of path-connectedness has an obvious analog for groupoids: A groupoid G is called *connected* if $G_{a,b} \neq \varnothing$ for all objects a,b of G. With this terminology we have:

Theorem 2.6. *Let $p: H \to G$ be a covering of groupoids and $f: A \to G$ a morphism from another groupoid A. Assume that all three groupoids are connected. Let x be an object of A, put $z = f(x)$, and pick any $y \in p^{-1}(z)$. Then the following two properties are equivalent:*

a) There is a morphism $g: X \to H$ of groupoids with $p \circ g = f$ and $g(x) = y$.

b) We have $f_*(A_x) \subset p_*(H_y)$ inside G_z.

Moreover, if these two equivalent conditions hold, there is a unique such g.

Proof. ...to be continued.

Chapter IV
Homology groups

1 Simplicial homology

2 Singular homology

3 Relative homology and excision

4 Comparison between simplicial and singular homology

5 The Eilenberg-Steenrod axioms

Chapter V

Appendix: Some homological algebra