Physics Beyond the Standard Model — Exercise Sheet 3

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Quantum Field Theory Summary

Cross sections, decay rates

The S-matrix is a central quantity of any QFT. For a general process $i \to f$ one can write the S-matrix element as $S_{fi} = \delta_{fi} + i(2\pi)^4\delta^{(4)}(p_i - p_f)T_{fi}$, with the T-matrix element T_{fi} and the total initial and final state momenta $p_{i,f}$. A general experimental question is: given some prepared initial state, what is the probability/the rate for a certain final state. One might be interested in a final state prescribed in detail, with definite spin configurations and/or momenta, or one might be interested in the sum/integral over a range of final state spins/momenta. Useful quantities to express such probabilites are cross sections and decay rates, which depend only on interesting physics and not on experimental details such as beam profiles etc. Decay rates and cross sections can be obtained from a given T-matrix element as follows.

Cross sections are defined for processes with two initial state particles and n final state particles with momenta $p_{A,B}$ and $p_1 \dots p_n$, respectively. The cross section depends on the infinitesimal final state phase space element, which we define first:

$$d\Phi_n := (2\pi)^4\delta^{(4)}(p_A + p_B - p_1 - \dots - p_n)d\tilde{p}_1 \dots d\tilde{p}_n, \tag{1}$$

$$d\tilde{p}_k := \frac{d^3p_k}{(2\pi)^3 2\sqrt{m_k^2 + \vec{p}_k^{\,2}}} = \frac{d^4p_k}{(2\pi)^4} 2\pi\delta({p_k}^2 - m_k^2). \tag{2}$$

Here the usual Lorentz-invariant integration measure appears, written in two different ways which are both useful depending on the context. The infinitesimal cross section for the scattering into this final state phase space element is given by

$$d\sigma = \frac{|T_{fi}|^2}{4\sqrt{(p_A \cdot p_B)^2 - m_A^2 m_B^2}} d\Phi_n \tag{3}$$

Note that often the initial state masses can be neglected, in which case the denominator simply becomes $2s$ with the Mandelstam variable $s = (p_A + p_B)^2$, such that

$$d\sigma = \frac{|T_{fi}|^2}{2s} d\Phi_n \qquad (m_{A,B} \to 0). \tag{4}$$

The decay rate for the decay of one initial state particle with mass m_A into the infinitesimal final state phase space element is given by

$$d\Gamma = \frac{|T_{fi}|^2}{2m_A} d\Phi_n. \tag{5}$$

A useful result for a simple phase space integral is

$$\int d\Phi_2 = \frac{1}{8\pi}. \tag{6}$$

External wave functions, spinors and γ-matrices

Feynman rules for S-matrix elements involve rules for "wave functions" of incoming and outgoing particles. For scalars the wave functions are simply unity, but for spinors and vectors, the wave functions are nontrivial. The rules are as follows:

$$\text{incoming/outgoing scalar} \qquad 1 \tag{7}$$
$$\text{incoming Dirac fermion} \qquad u(p,s) \tag{8}$$
$$\text{outgoing Dirac fermion} \qquad \bar{u}(p,s) \tag{9}$$
$$\text{incoming Dirac anti-fermion} \qquad \bar{v}(p,s) \tag{10}$$
$$\text{outgoing Dirac anti-fermion} \qquad v(p,s) \tag{11}$$
$$\text{incoming vector} \qquad \epsilon^{\mu}(p,s) \tag{12}$$
$$\text{outgoing vector} \qquad \epsilon^{\mu *}(p,s) \tag{13}$$

Here p and s always refer to the respective particle momentum and spin. The spinors u and v satisfy the following Dirac equations and sum rules:

$$(\not{p} - m)u(p,s) = 0 \qquad \sum_s u(p,s)\bar{u}(p,s) = \not{p} + m \tag{14a}$$
$$(\not{p} + m)v(p,s) = 0 \qquad \sum_s v(p,s)\bar{v}(p,s) = \not{p} - m \tag{14b}$$

The polarization vectors $\epsilon^{\mu}(p,s)$ and the range of possible polarizations depends on the gauge choice. Physical polarizations of massless photons must be transverse; a massive vector boson has three physical spin orientations, but often one can use the simple sum rule

$$\sum_s \epsilon^{\mu}(p,s)\epsilon^{\nu *}(p,s) = -g^{\mu\nu}. \tag{15}$$

After squaring a T-matrix element and summing/averaging over spins, often traces of γ-matrices or $\not{p}$-objects appear. Simple trace formulas are

$$\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) = 4g^{\mu\nu}, \tag{16}$$
$$\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}), \tag{17}$$
$$\mathrm{Tr}(\not{p}_1\gamma^{\mu}\not{p}_2\gamma^{\nu}) = 4(p_1^{\mu}p_2^{\nu} + p_2^{\mu}p_1^{\nu} - g^{\mu\nu}p_1\cdot p_2), \tag{18}$$

where the last formula is a useful special case of the second.

1. **Cross sections and decay widths in the Z' extension**

Consider a Z' model with a Z' mass M and a Z' propagator including higher order corrections given by the denominator $D(p^2) = 1/(p^2 - M^2 + iM\Gamma)$.

Assume the following Z' interaction with a quark q and another fermion f:

$$\mathcal{L}_{\text{int}} = -g_q \bar{q}\gamma^\mu q Z'_\mu - g_f \bar{f}\gamma^\mu f Z'_\mu \, .$$

a) Compute the T-matrix element for the process $Z' \to f\bar{f}$, square it an sum over final state spins and average over initial state spins. Note the hint in part d). Then show that

$$\Gamma_{Z' \to f\bar{f}} = \int d\Gamma = \frac{g_f^2 M}{12\pi} \, .$$

b) Compute the T-matrix element for the process $q\bar{q} \to Z'$, square it an sum over the final state spin and average over the initial state spins. Then show that

$$\sigma_{q\bar{q}\to Z'} = \int d\sigma = g_q^2 \pi \delta(s - M^2).$$

c) Compute the T-matrix element for the process $q\bar{q} \to f\bar{f}$, square it an sum over final state spins and average over initial state spins. Then show that

$$\sigma_{q\bar{q}\to f\bar{f}} = \int d\sigma = \frac{1}{12\pi} s |D(s)|^2 g_q^2 g_f^2 \, .$$

d) A useful way to organize the calculation is to define as an intermediate quantity the "fermionic tensor"

$$T_f^{\mu\nu} := \sum_{s_1,s_2} \bar{u}(p_1,s_1)\gamma^\mu v(p_2,s_2)\bar{v}(p_2,s_2)\gamma^\nu u(p_1,s_1).$$

Compute it using Casimir's trick (reorder such that you get a trace and are able to use the sum rules in Eq. (14)).

Then show that the "Ward identity"

$$(p_1 + p_2)_\mu T_f^{\mu\nu} = 0$$

holds. This Ward identity is the reason why the replacement (15) is correct since all other terms which could exist on the right-hand side drop out.