3.3 Phenomenological Neuron Models

Learning objectives:

In this chapter I would like to introduce your phenomenological neuron models. these form a bridge between the highly simplified binary models and the detailed realistic models of neurons, which we will discuss in the following section.

In this chapter I would like to answer the following questions:

- What is an leaky integrate and fire neuron model (LIF Neuron) and how it works?
- What does the introduction of a refractory period in the model do?
- How does the spiking frequency depend on the input current in LIF neurons?
- What advantages do you get when you switch to non-linear models IF models?
- What is adaptation and how is it modeled?
- What is the benefit of phenomenological neuron models?

3.3 Phenomenological Neuron Models

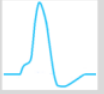
- In the last chapter we dealt with a rather simple neuron model, which is extremely common and is used a lot in modern AI algorithms.
  - However, it has only little biological properties, in particular there is no time dependency here.
  - Now this may not seem so tragic. Think of flying as an example: today we can fly (to almost every corner of the world) even though we don't imitate bird flight. This has been tried for hundreds of years and all attempts have failed.
  - So you can say that to imitate the biology we don't have to replicate it one to one. The argument is correct and is really important to note, but with the MCP neuron we cannot draw any conclusions about any cellular processes. However, this is important if you want to understand a little more about biology.
- So if we want to do this, we need models to reproduce the important properties of the neurons.
  - Not in detail but on a phenomenological level. We now want to look at these models.
  - Models that builds a bridge between detailed models (let's look at them later) and the simple digital models from this last chapter.

Phenomenological Models

Binary Neuron, Standard in Neuroinformatics
- simple implementation
- time independent

Leaky Integrate and Fire Neuron (LIF)
- time dependent
- refractory period is implementable
- linear integration behavior

Nonlinear integrate-and-fire model
- time dependent
- refractory period is implementable
- self-integration
- supralinear integration behavior



Biological Neuron (sketch)
- complicated: e.g. time dependence, self-integration, adaption, etc.

3.3 Phenomenological Neuron Models

- Let's start our discussion with the following two slides that we discussed in chapter 2.1:

➢ Each presynaptic spike t_j^f evokes an excitatory postsynaptic potential (EPSP) $u_i(t)$ that can be measured with an electrode as a potential difference:

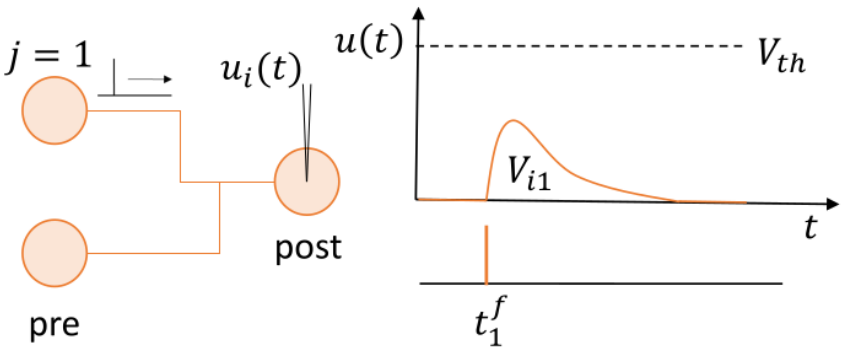
$$u_i(t) - u_{rest} = V_{ij}(t)$$

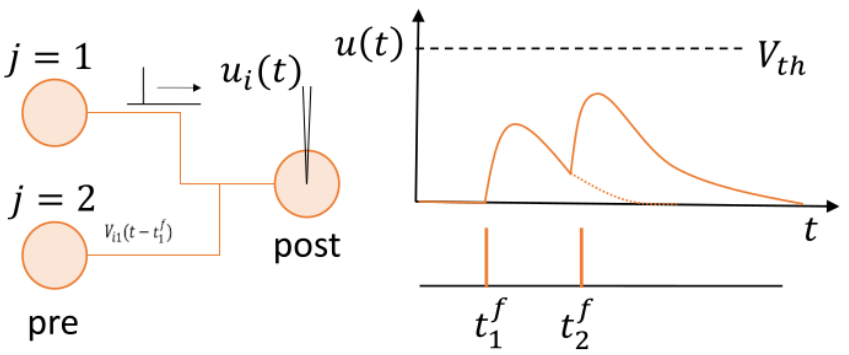
➢ The time course of the EPSP caused by the spike of neuron is:

for j = 1 : $V_{i1}(t - t_1^f)$

➢ An input spike from a second presynaptic neuron j = 2 that arrives shortly after the spike from neuron j = 1 causes a second postsynaptic potential that adds to the first one:

$$u_i(t) = \sum_j \sum_f V_{ij}(t - t_j^f) + u_{rest}$$

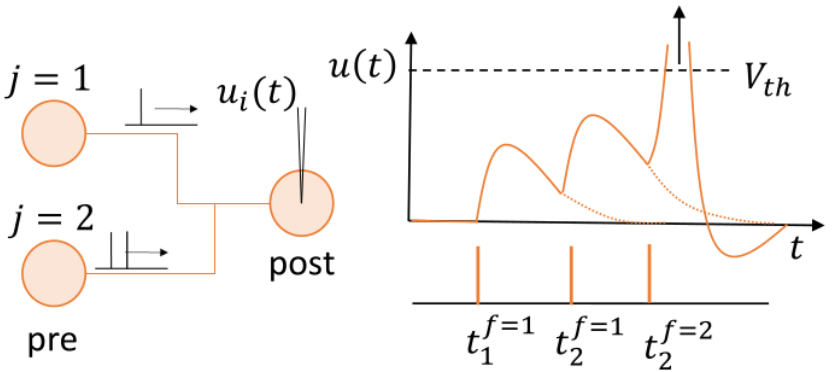




3.3 Phenomenological Neuron Models

- If $u_i(t)$ reaches the threshold V_{th}, an action potential is triggered.

$$u_i(t) = V_{th} \quad \text{and} \quad \frac{du_i(t)}{dt} > 0 \Rightarrow t = t_j^f$$



- As a consequence, the membrane potential starts a large positive pulse-like excursion (arrow). On the voltage scale of the graph, the peak of the pulse is out of bounds.
- After the pulse the voltage returns to a value below the resting potential.

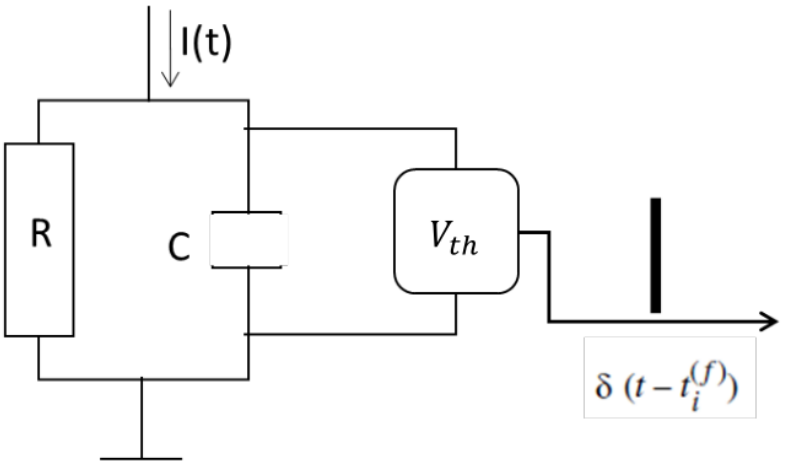
- We have already converted the essential process at the synapse into a (technical) mathematical model.
  - Although this is quite simple and only describes the biology in a rudimentary manner, it already includes the neuronal dynamics.

- From this mathematical point of view, it is not a big thing to design a small circuit model for the neuron.
  - From now on something new is coming. So watch out again!

3.3 Phenomenological Neuron Models

- Designing a circuit for this is quick for an electrical engineer, especially since we already know part of the circuit:
  - So we have an *RC*-integrator to emulate cellular integration (membrane potential), which integrates an input current *I(t).*
  - At the same time, there is also a threshold value switch with a pulse generator that always emits a pulse when the threshold value V_{th} is reached:

$$u_i(t) = V_{th} \quad \text{and} \quad \frac{du_i(t)}{dt} > 0 \Rightarrow t = t_j^f$$



  - We will look in detail later at how exactly this threshold electronics is implemented technically in a circuit. Here we want to leave that on that abstract level. Next we can derive the equation for this circuit:

$$I(t) = I_R + I_{\text{cap}} \quad \Rightarrow \quad I(t) = \frac{u(t)}{R} + C\,\frac{\mathrm{d}u}{\mathrm{d}t} \quad \Rightarrow \quad \boxed{\tau_m\,\frac{\mathrm{d}u}{\mathrm{d}t} = -u(t) + R\,I(t)} \qquad \text{with} \quad \tau_m = R\,C$$

$$I_R = u/R \qquad C = q/u \;\Rightarrow\; I_{\text{cap}} = C\,\mathrm{d}u/\mathrm{d}t$$

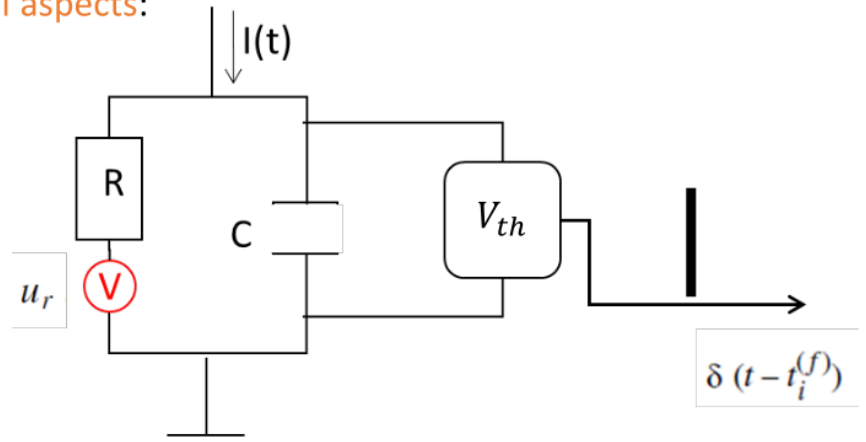
  - Now this is the simplest version of the **leaky integrate and fire neuron model (LIF Neuron)**.

3.3 Phenomenological Neuron Models

- Now this simple model can be expanded to include other biological aspects:
  - Neurons are active elements, i.e. the **resting potential** is at a negative level.
  - We can introduce this resting potential by introducing a voltage source:
    - In the differential equation $\tau_m \frac{\mathrm{d}u}{\mathrm{d}t} = -u(t) + R\,I(t)$

      we replace the membrane voltage $u(t)$ by a constant offset voltage u_r -called resting potential:

      $$u(t) \rightarrow u(t) - u_r$$



  - Where we place the following condition on the membrane potential: $\lim_{t \rightarrow t^{(f)}; t > t^{(f)}} u(t) = u_r$

  - Another important property is the **refractory period**, which is of crucial importance. No further action potential can be generated within the time span Δ^{abs} and thus there can be no permanent excitement and there is a maximum pulse frequencies:

$$u(t) = u_r \quad \text{for} \quad t = t^{(f)} + \Delta^{abs}$$

3.3 Phenomenological Neuron Models

- For a better understanding, we consider two examples below:
- **Example 1:** *constant stimulation and firing rates*
  - constant input current: $I(t) = I_0$
  - Boundary conditions:
    - First spike time $t^{(f=1)} = t^{(1)}$
    - $u_r = 0$
    - $u\left(t^{(1)}\right) = u_r = 0$

$$\tau_m \frac{du}{dt} = -u(t) + R\,I(t) \quad \Rightarrow \quad u(t) = R\,I_0\left[1 - \exp\left(-\frac{t - t^{(1)}}{\tau_m}\right)\right]$$

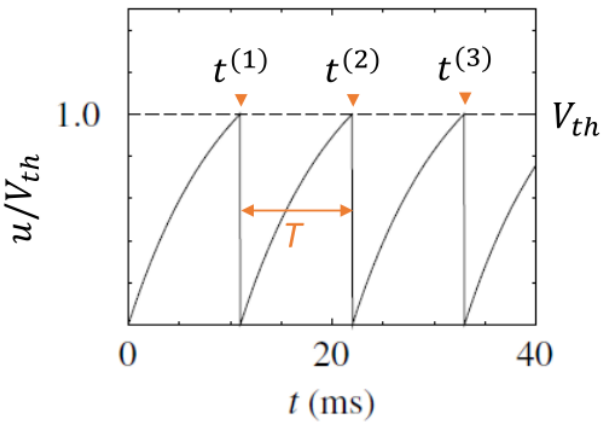
  - Solving the differential equation was not a big deal, but let's take a closer look at the solution using the following 3 cases:

[Case 1] for $t_0 \to \infty$ ➡ $u(\infty) = RI_0$

[Case 2] $RI_0 < V_{th}$ ➡ no spike

[Case 3] $RI_0 > V_{th}$ ➡ u(t) reaches threshold at $t^{(2)}$

$$V_{th} = R\,I_0\left[1 - \exp\left(-\frac{t^{(2)} - t^{(1)}}{\tau_m}\right)\right]$$



➢ for a constant input current I_0, the LIF neuron fires regularly with period T :

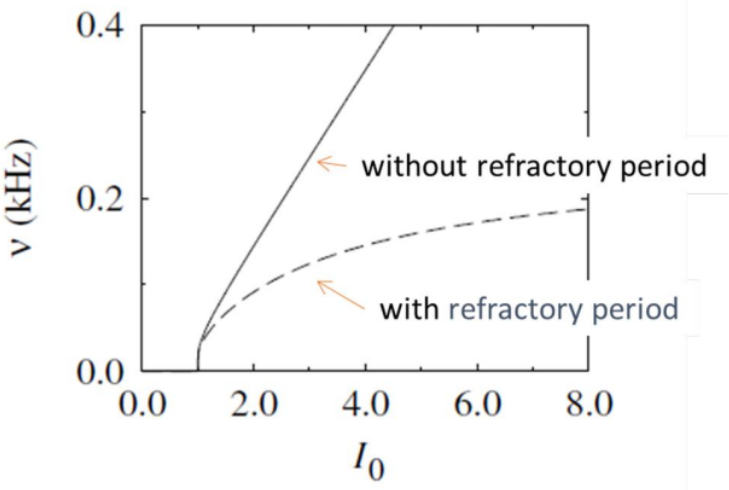
$$T = t^{(2)} - t^{(1)}$$

3.3 Phenomenological Neuron Models

- Taking into account a refractory period, the average rate of fire can also be determined:
  - firing period including refractory time: $T' = T + \Delta^{abs}$
  - Where the firing rate (frequency) is: $\upsilon = \frac{1}{T'}$

$$\nu = \left[\Delta^{\text{abs}} + \tau_m \ln \frac{R\, I_0}{R\, I_0 - \vartheta} \right]^{-1}$$

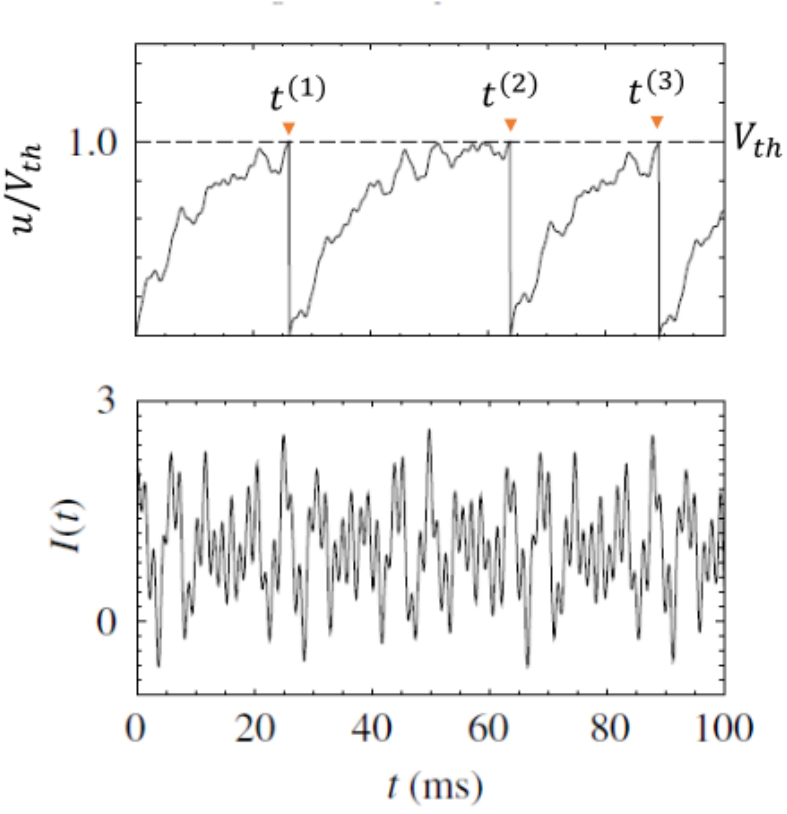
- if you plot the frequency as a function of the constant input current, you can clearly see the influence of the refractory period.
- the frequency saturates to a maximum frequency value. So exactly as required in biology.



3.3 Phenomenological Neuron Models

- As a second example, we consider an input current that varies over time. In principle, you can imagine any time-dependent current here. For the example, we simply assume a current with constant noise, as shown in the figure below:
- **Example 2:** *time-dependent stimulus I(t)*
  - noisy input current: $I = I(t)$

$$\tau_m \frac{du}{dt} = -u(t) + R\,I(t) \quad \Rightarrow \quad u(t) = u_r \exp\left(-\frac{t-\hat{t}}{\tau_m}\right) + \frac{1}{C}\int_0^{t-\hat{t}} \exp\left(-\frac{s}{\tau_m}\right) I(t-s)\,ds$$



  - $\hat{t}$ are the spike time of the respective last spike $\hat{t} = t^{f-1}$.
  - since I(t) is time-dependent, this can no longer be solved analytically. But numerically this is not a problem and the solution is shown in the figure.
  - Basically we have a similar behaviour as in example 1, but as can be seen well, the firing times vary somewhat due to the varying input current.
  - We will do a more detailed look at the LIF neuron in the tutorial.

3.3 Phenomenological Neuron Models

- The LIF neuron has already the essential property of a neuron, such as a threshold for the activity, integration of input signals, refractory time and frequency coding.
  - However, it does not include strong nonlinearities like real neurons.
  - The effect of self-integration should be mentioned here, i.e. after a certain point there is no going back.
  - To take such non-linear effects into account, we first generalize the simple LIF differential equation to the form:

$$\tau_m \frac{du}{dt} = -u(t) + R\,I(t) \quad \Rightarrow \quad \boxed{\tau \frac{du}{dt} = F(u) + G(u)\,I}$$

  - here are *F(u)* and *G(u)* are generalized terms for the voltage and resistance dependencies.

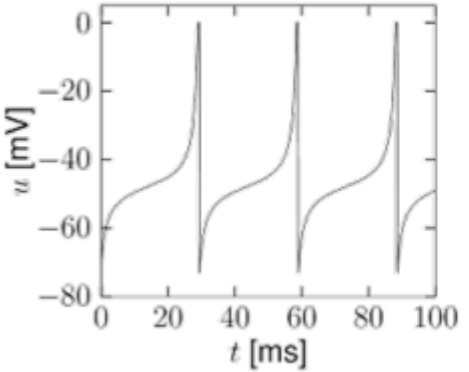
- The following quadratic form for *F(u)* can be chosen mathematically,

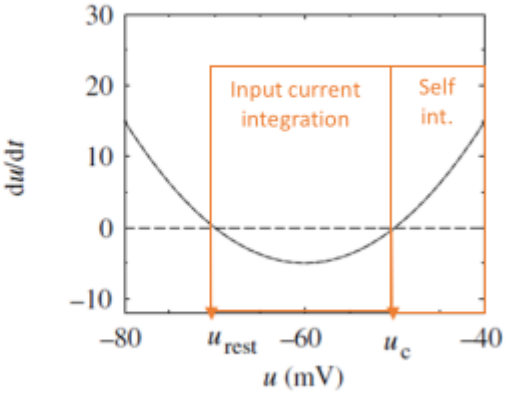
$$F(u) = a_0\,(u - u_{\text{rest}})\,(u - u_c) \quad \Rightarrow \quad \boxed{\tau \frac{du}{dt} = a_0\,(u - u_{\text{rest}})\,(u - u_c) + RI}$$

  - With a_0 a positive constant and u_c the critical voltage for the self-integration.
  - This neuron model is known as **quadratic integrate and fire neuron (QIF neuron)**.

3.3 Phenomenological Neuron Models

- The main characteristics of the QIF neuron can be seen in the figures on the right.
- The non-linearity of the model can be clearly seen in the *u*-*t* characteristic.
- In the second figure d*u*/d*t* is plotted as a function of *u*: this shows the self-integration from the voltage u_c: d*u*/d*t* > *0*



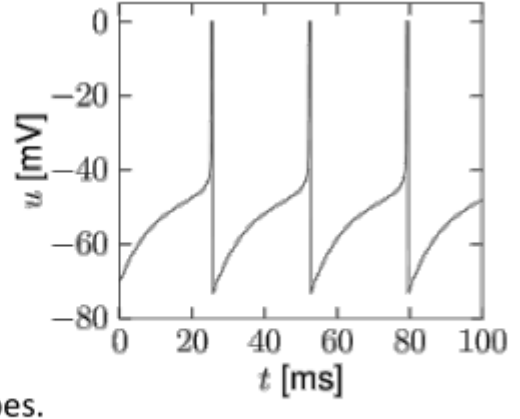


- Another non-linear variant is the **exponential integrate-and-fire model** , which, as the name suggests, uses an exponential voltage dependency:

$$F(u) = -(u - u_{\text{rest}}) + \Delta_T \exp\left(\frac{u - \vartheta_{rh}}{\Delta_T}\right)$$

- The differential form is used to obtain the following form:

$$\tau \frac{\mathrm{d}}{\mathrm{d}t} u = -(u - u_{\text{rest}}) + \Delta_T \exp\left(\frac{u - \vartheta_{rh}}{\Delta_T}\right) + RI$$



- Here Δ_T and ϑ_{rh} are fit parameters that can be adapted to the desired curve shapes.
- In limit case $\Delta_T \to 0$, the model merges into the LIF neuron model.

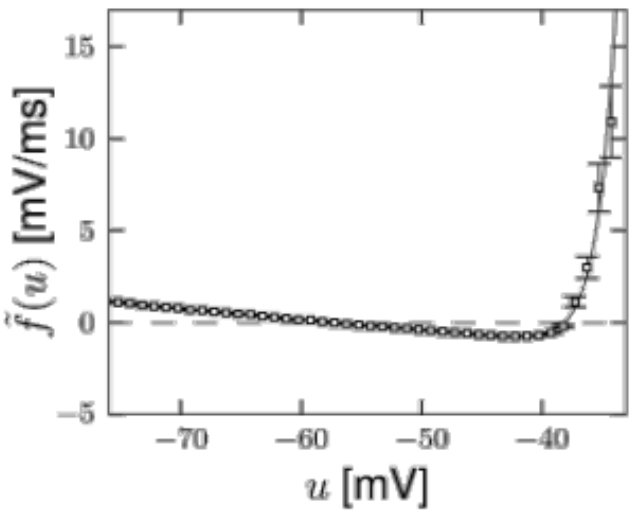
3.3 Phenomenological Neuron Models

- Now we turn to the question to what extent do these neuron models describe those spike patterns that can be measured in real neurons?
  - For comparison to experimental physiological data, we rewrite the general form of the leaky-integrate and fire neuron as follows:

$$\tau \frac{\mathrm{d}}{\mathrm{d}t} u = f(u) + R(u)\, I \qquad \tilde{f}(u) = f(u)/\tau \qquad \tilde{f}(u(t)) = \frac{1}{C} I(t) - \frac{\mathrm{d}}{\mathrm{d}t} u(t)$$

  - For the exponential integrate-and-fire neuron model you get the following expression, which you can start with real characteristics of neurons:

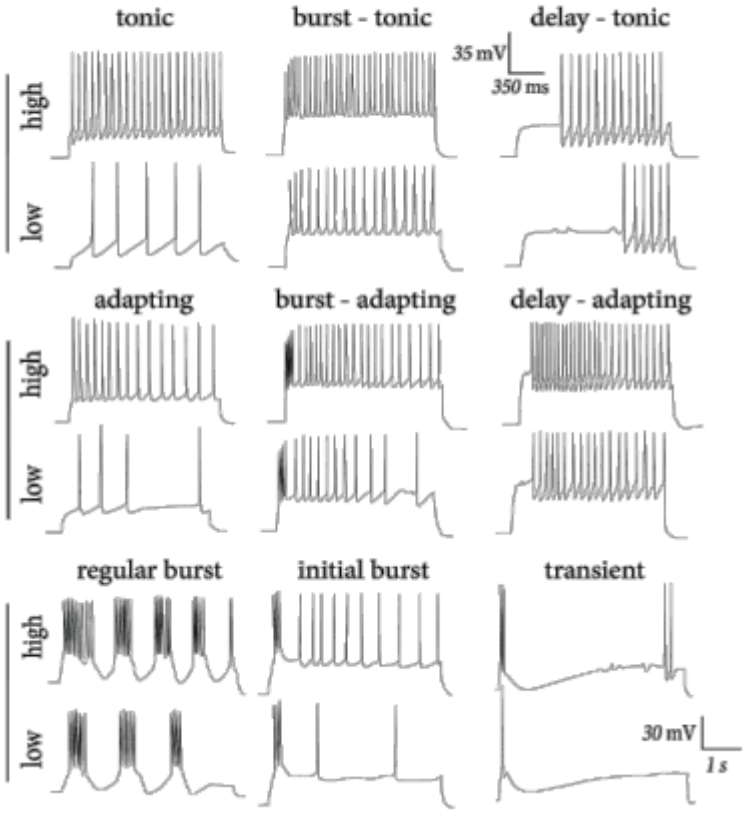
$$\tilde{f}(u) = -\frac{u - u_{\text{rest}}}{\tau} + \frac{\Delta_T}{\tau} \exp\left(\frac{u - \vartheta_{rh}}{\Delta_T}\right)$$



  - The figure shows the comparison of real neuron data (dots) and model (solid line). As you can see, the line shape can be reproduced quite well with the model.

3.3 Phenomenological Neuron Models

- Now we already have a very simple model which can already emulate the characteristics of a real action potentials.
  - What we used to get this is actually not much - we have added a nonlinearity to the simple RC circuit for emulating the cell membrane.
- The question remains is it all? – or is it enough to emulate the dynamics of a neuron?
  - Unfortunately not yet and we have to take a closer look at the different spike patterns that occur.
  - Such an overview is shown in the figure.
  - What is shown are spike trains recorded for cortical neurons under an high and low constant stimulation.
  - What you can see is a variety of different pattern which ranges from regular spiking to bursting. Further you see a delay and nearly in all spike trains adaptation.
  - Adaptation means that the spike frequency is decreasing by time.
  - Thus, those different spike pattern we seek to into our neuron model…. Let's dot it .



3.3 Phenomenological Neuron Models

- As we have seen, the voltage curve of the membrane potential can be simulated well using the function *f(u)* in the context of the nonlinear integrate-and-fire neuron model.
  - However, a single differential equation is not sufficient to simulate the different spike patterns, and an additional equation is required here that describes the dynamics of individual variables.
  - The effect of the adaptation can be simulated by the following set of differential equations:

$$\tau_m \frac{\mathrm{d}u}{\mathrm{d}t} = f(u) - R\sum_k w_k + R\,I(t)$$
$$\tau_k \frac{\mathrm{d}w_k}{\mathrm{d}t} = a_k\,(u - u_{\mathrm{rest}}) - w_k + b_k \tau_k \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

  - a_k and b_k are positive constants and τ_k is the time constant of the process.
  - Thus, those set of equation couples the voltage equation to an abstract current variables $\omega_{k,}$ which influences the input current for the integration in the voltage equation depending on the voltage curve. With increasing ω_k the influence of u becomes less and less. The third term with delta function changes the dynamic of ω_k additionally at every spike time.
  - The variable ω_k -if you only consider the first two terms-it is easy to see that you have integrated the effect of the adaptation into the model.

3.3 Phenomenological Neuron Models

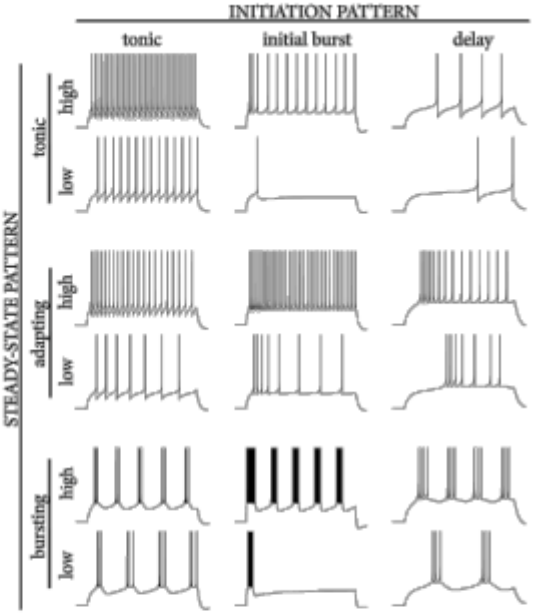
- If one chooses the exponential integrate-and-fire model, i.e.

$$f(u) = -(u - u_{\text{rest}}) + \Delta_T \exp\left(\frac{u - \vartheta_{rh}}{\Delta_T}\right)$$

  - then you get the following system on coupled differential equation named as **Adaptive Exponential Integrate-and-Fire neuron (AdEx):**

$$\tau_m \frac{\mathrm{d}u}{\mathrm{d}t} = -(u - u_{\text{rest}}) + \Delta_T \exp\left(\frac{u - \vartheta_{rh}}{\Delta_T}\right) - Rw + RI(t)$$
$$\tau_w \frac{\mathrm{d}w}{\mathrm{d}t} = a(u - u_{\text{rest}}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right).$$

  - With a suitable choice of parameters, the spike patterns shown in the figure can be generated.



- Alternatively, the QIF model can also be used, which gives the following set of differential equation, referred to as Izhikevich model:

$$\tau_m \frac{\mathrm{d}u}{\mathrm{d}t} = (u - u_{\text{rest}})(u - \vartheta) - Rw + RI(t)$$
$$\tau_w \frac{\mathrm{d}w}{\mathrm{d}t} = a(u - u_{\text{rest}}) - w + b\tau_w \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

  - If $u = \vartheta_{reset}$, the voltage is reset to $u=u_r$ and the adaptation variable w is increased by an amount b. (Normally b is positive, but $b < 0$ is also possible.)

3.3 Phenomenological Neuron Models

- Hopefully, I was able to convince them that by choosing a non-linear leaky integrate and fire model you can recreate all of the spike pattern characteristics.
  - These models therefore show a much better agreement with biology than the MCP neuron.
  - However, this requires two coupled differential equations.
  - The use of such a model in larger networks makes it significantly more computationally intensive than using the MCP neuron.
- At this point it is easy to underpin where neuromorphic systems offer advantages to purely algorithmic (mathematical) models.
  - They allow the simulation of more biologically realistic models in hardware with significantly less computing effort.
  - We want to take a closer look at this in the following using an example.
  - The neuron model shown comes from a former doctoral student of mine -Marina Ignatov- which is why we named it *the Ignatov Neuron**

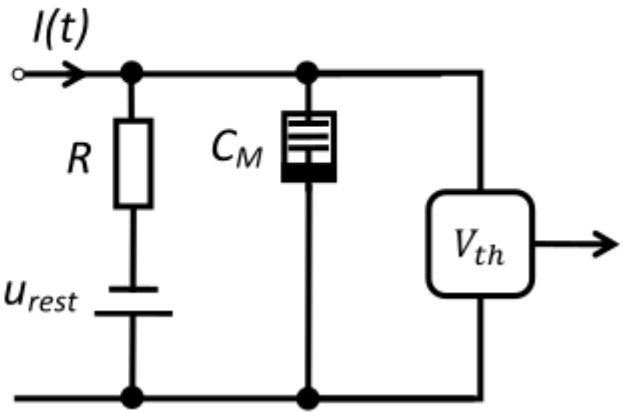
*Ignatov M., Ziegler M., Front. Neurosci. 9:376. doi: 10.3389/fnins.2015.00376.

3.3 Phenomenological Neuron Models

- For this we consider the circuit shown in the figure:
  - This consists of a resistor and a new, novel component, a MemCapacitor.
  - A MemCapacitor is a capacitance with memory, i.e. it changes the capacity depending on the charge that has flowed through it.
  - This can be described by the following set of differential equations:

$$Q(t) = C(\omega) \cdot u(t) \quad \text{and} \quad \frac{d\omega(t)}{dt} = \beta\, \omega(t) \cdot I(t)$$

ω is a state variable that changes under current flow. β is a material-specific constant. We will discuss this device later in detail and should not stop us...



- What is interesting here is what does such a component do in the LIF neuron differential equation:

$$\frac{du}{dt} = \frac{1}{R\,C(\omega)} f(u) + \frac{1}{C(\omega)} I(t)$$

  - As you can see, the MemCapicator affects the time constant for integration. As a result, the integration time is gradually increased due to an increasing capacity value.
  - Thus, this leads to a reduction in the spike frequency. So exactly what you see in adaptation.

3.4 Biological Neuron Models

Learning objectives:

In this chapter I would like to introduce you to neuron models that allow to model the bio-chemical processes that take place in the nerve cells to generate an action potential. For this I present the Hodgkin and Huxley neuron model to you. As a further goal of this chapter, I would like to show how you can get from this rather complex model to more manageable models, which still allow conclusions to be drawn about the parameters for describing the ion dynamics. In the end I present an implementation within an electronic circuit to build a bridge to neuromorphic engineering.

In this chapter I would like to answer the following questions:

- What is the Hodgkin and Huxley neuron model?
- What are the gating variables and what do they describing?
- What is the FitzHugh-Nagumo neuron model?
- How works the phase plane analysis and what is a relaxation oscillator?
- How to solve the van der Pol oscillator?

3.4 Biological Neuron Models

- In the last chapter we showed that the following set of differential equations is sufficient to mathematically simulate the dynamics and behaviour of neurons.
  - In general terms we came to the general nonlinear integrate-and-fire model:

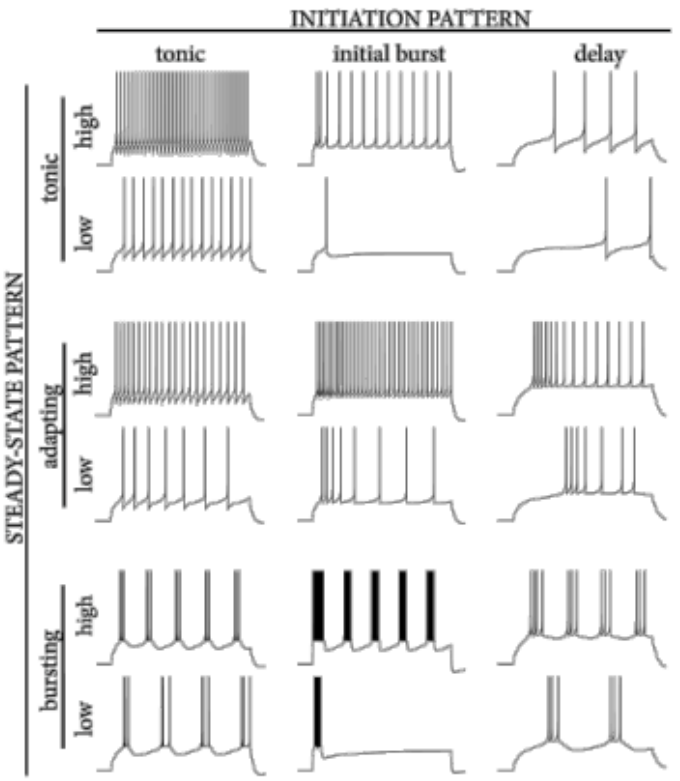
$$\tau \frac{\mathrm{d}u}{\mathrm{d}t} = f(u) + G(u)\, I$$

  - We were able to show that by choosing a suitable non-linearity for *f(u)* and by using a second differential equation, the complete behaviour of a neuron can be simulated (see figure):

$$\tau_m \frac{\mathrm{d}u}{\mathrm{d}t} = f(u) - R \sum_k w_k + R\, I(t)$$

$$\tau_k \frac{\mathrm{d}w_k}{\mathrm{d}t} = a_k \left(u - u_{\mathrm{rest}}\right) - w_k + b_k \tau_k \sum_{t^{(f)}} \delta\left(t - t^{(f)}\right)$$

  - However, this model does not allow any conclusions to be drawn about the bio-chemical processes that take place in the cell.

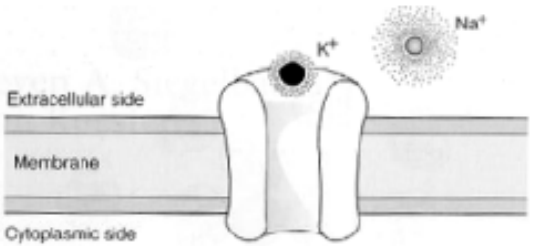


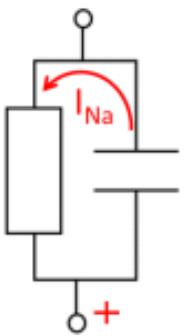
We therefore want to look at an expansion of the methodology for emulating the biochemical processes in this chapter

3.4 Biological Neuron Models

- Already in 1952, Hodgkin and Huxley examined the development of action potentials on the giant axon of a squid and, in addition to their measurements, designed a physical model, the so-called Hodgkin and Huxley neuron model.
  - In 1962 the two scientists received the Nobel Prize for their work.
  - The model originally describes three types of ion channels, but can be expanded to include more.

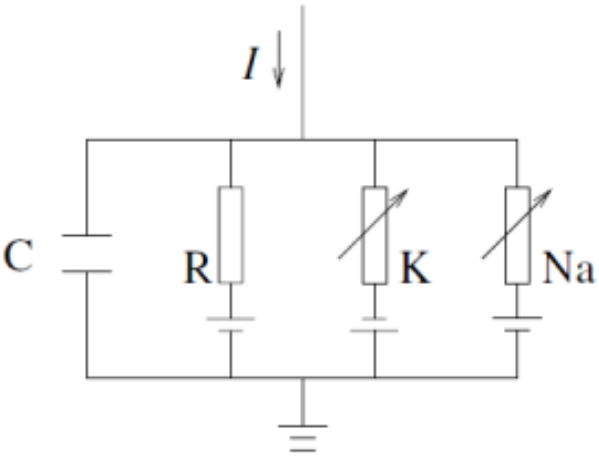
- Before we look at the model in detail, we should recall something about the generation of an action potential from chapter 2:
  - the origin of action potentials lies in the special property of the voltage dependency of ion channels: in response to a membrane polarization, sodium channels can open.
  - If a threshold value is reached during the depolarization of the membrane potential, ion channels open for Na^+ ions.
  - The more positive the membrane potential becomes, the more ion channels open.
  - When the membrane potential reaches about +50 mV, the ion channels close again and the cell interior is positively charged to the cell exterior.
  - This has an effect on K^+-ions, which are transported out of the cell, making the cell interior negative again.
  - This is easy to understand our small RC circuit: assuming the capacitor is charged, when the potential at the negative pole increases, the capacitor is discharged via the resistor.





3.4 Biological Neuron Models

- The Hodgkin and Huxley neuron model can be represented within a simple circuit model as shown in the figure:
  - According to the slide before capacitor acts as the cell membrane and separates the interior of the cell from the extracellular liquid.
  - If an input current I(t) is injected into the cell, the current either charges the capacitor or flow through the ion channels denote as an variable resistance R_K and R_{Na} in the model
  - Further the model accounts for a leakage channel/resistance R.
  - The battery (defined resting potential) accounts the fact that of active ion transport through the cell membrane, the ion concentration inside the cell is different from that in the extracellular liquid.
  - As the different ion channels have different resting potentials, one battery is uesd for each channel.



- In analogy to the LIF neuron, the circuit can easily be transferred into a mathematical equation:

$$I(t) = I_{\text{cap}}(t) + \sum_k I_k(t)$$

$C = Q/u$ and $I_{\text{cap}} = C\,\mathrm{d}u/\mathrm{d}t$.

$$C\frac{\mathrm{d}u}{\mathrm{d}t} = -\sum_k I_k(t) + I(t)$$

3.4 Biological Neuron Models

- So far that's nothing great new. It is a bit like the one in the previous chapter. And yes the procedure is similar. Next you need the voltage dependence of the ion channels:
  - the sumof the ionic currents which pass through the cell membrane $\sum_k I_k$ must be rewritten by their voltage dependency.
  - This is exactly what Hodgkin and Huxley did and they found: $$\sum_k I_k = g_{\mathrm{Na}}\, m^3 h\, (u - E_{\mathrm{Na}}) + g_{\mathrm{K}}\, n^4\, (u - E_{\mathrm{K}}) + g_{\mathrm{L}}\, (u - E_{\mathrm{L}})$$

- This equation is somewhat complicated and requires closer examination.
  - But before we do so let's define the different variables:
  - The leakage channel is described by a voltage-independent conductance $g_L = \frac{1}{R}$.
  - Since u is the total voltage across the cell membrane and E_L the voltage of the battery (reverse potentials):
$$\Rightarrow I_L = g_L(u - E_L)$$
  - The conductance of the channels is voltage and time dependent. Thus, if all channels are open, they transmit currents with a maximum conductance g_K and g_{Na}.
  - Parameters for the Hodgkin-Huxley equations fitted on pyramidal neurons of the cortex are shown in the table.

x	E_x [mV]	g_x [mS / cm^2]
Na	55	40
K	-77	35
L	-65	0.3

3.4 Biological Neuron Models

- The important thing about the Hodgkin and Huxley model is that it contains information about the effective resistance of the individual ion channels as function of time and voltage. For this, they introduces the gating variables m, n, and h; with which the channel flows can be written as follows:

$$\Rightarrow I_{Na} = g_{Na} m^3 h (u - E_{Na}) \quad \text{and} \quad I_K = g_K n^4 (u - E_K)$$

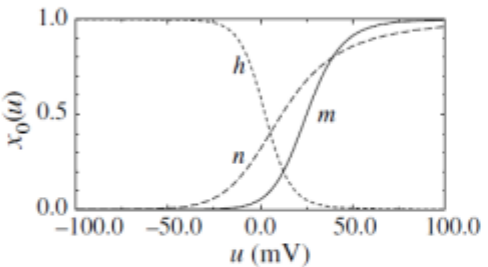
- Thus, the gating variables gives the probability that a channel is open at a given moment in time. The following functionalities are assigned to the variables in more detail:
  - The combined action of m and h controls the Na channels and the K channels are controlled by n.
  - m describes the activation (opening) of the Na channel and h its inactivation (blocking).
  - n describes the activation of the K channel.
- The three gating variables evolve according to differential equations of the form:

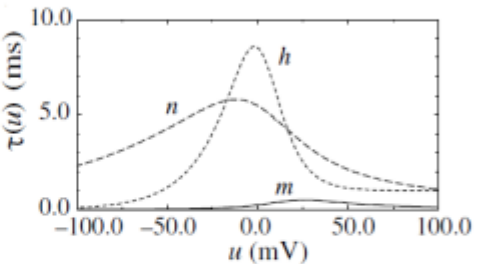
$$\dot{x} = \frac{dx}{dt} = -\frac{1}{\tau_x(u)}\left(x - x_0(u)\right) \qquad \text{for } x = m, h, or\ n$$

  - Here τ_x is the characteristic time constant of the respective ion channel, while x_0 is an asymptotic value that takes into account a maximum saturation value.

3.4 Biological Neuron Models

- We now have a system of differential equations that we can solve and we want to do this in the following. Since the solution is not really difficult but the equations are a bit long we want to proceed step by step.
  - Let's take a look at what the equation says: for a fixed value of u, the variable x approaches the value $x_0(u)$ with the time constant $\tau_x(u)$.
  - The voltage dependency of the two variables $x_0(u)$ and $\tau_x(u)$ is shown in the graphs on the right.
  - These were extracted from measurements of pyramidal neurons in the cortex and are therefore numerically determined functions.





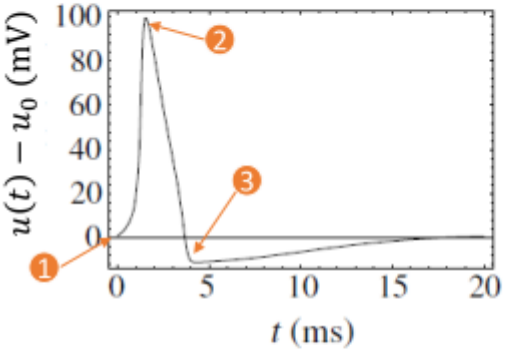
  - Let us now consider the example with the following boundary conditions: $u_0 = -65\ mV$ for $t < t_0$ and then switches to u. This can be used to calculate the equation for x and we get for the gating variables:

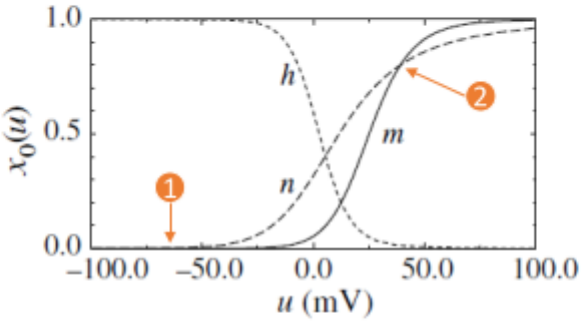
$$x(t) = x_0(u_1) + \left(x_0(u_0) - x_0(u_1)\right) \exp \frac{-(t - t_0)}{\tau_x(u_1)}$$

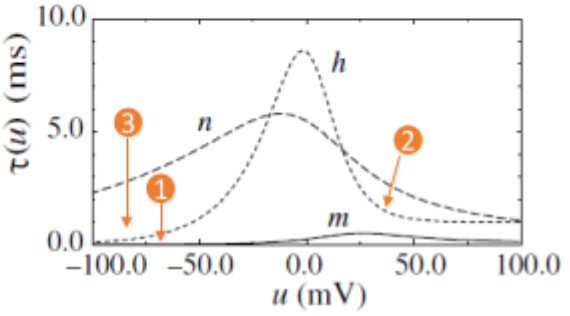
  - Thus with the use of the functions for x_0 and $\tau_x(u_1)$ we can calculate the sodium I_{Na} and potassium current I_K.

3.4 Biological Neuron Models

- Now, we have the necessary formulas together which enables us to understand the biophysics underlying the generation of an action potential:







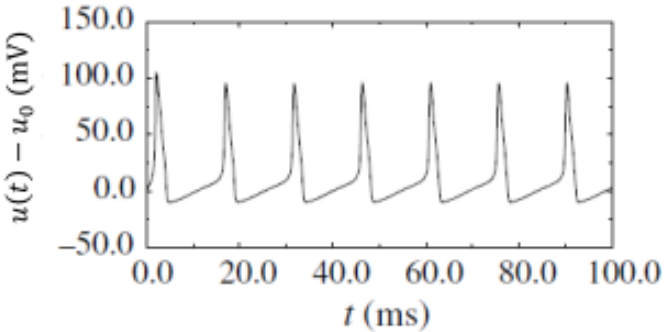
- The variables *n* and *m* increase and *h* decreases with increasing voltage *u* ($u_0 = -65\ mV$ which is labelled as 1).
- This leads to an opening of the ion channels and an action potential is initiated (**label 1**).
- At high values of *u*, the sodium conductance is slowly switched off by the factor *h:* τ_h is getting relatively small which leads to extreme short time constants in the exponential $\propto \exp\frac{-t}{\tau_h}$ and thus high impact of h (**label 2**).
- Since τ_h is always greater than τ_m, *h* reacts more slowly to a change in voltage than *m* the potassium (K^+) current sets in on a slow time scale: action potential followed by a negative overshoot (**label 3**).

3.4 Biological Neuron Models

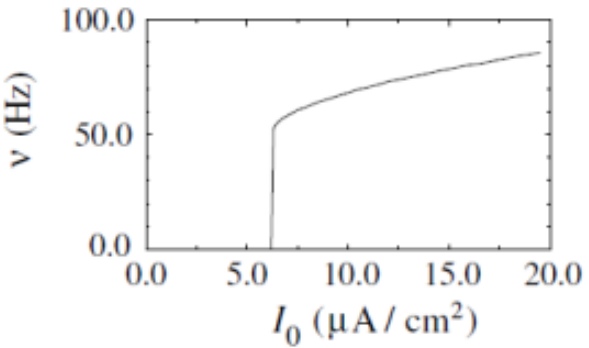
- In addition to generating the action potential, other important properties of neuronal excitation can also be demonstrated within the Hodgkin-Huxley model. A really important aspect is information coding:

$$C\frac{\mathrm{d}u}{\mathrm{d}t} = -\sum_k I_k(t) + I(t)$$

- If we consider a constant input current $I(t) = I_0$ for $t > 0$, we observe a constant spike pattern in the model for $I_0 \geq 2.7\ \mu m/cm^2$ (see figure on the right)



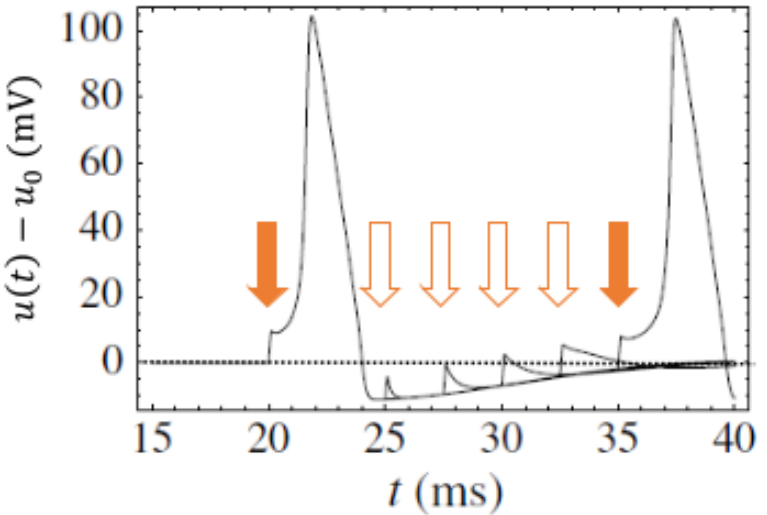
- This leads to the frequency-current representation shown in the figure below: firing rate $\nu = 1/T$ and T is the inter-spike interval.



- Now this should already be familiar to one or the other. If you do not know this, please look again in chapter 2 and find that this corresponds to the mean firing rate coding.

3.4 Biological Neuron Models

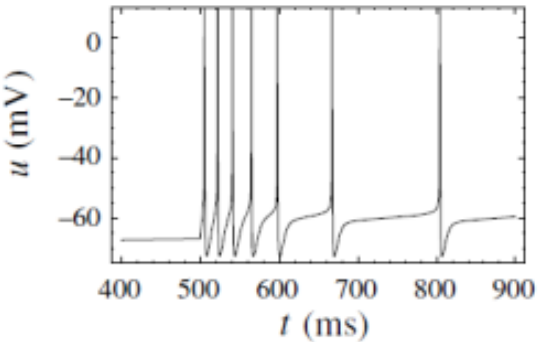
- Another important property of action potentials is the refractory period, which prevents two signals from becoming indistinguishable from each another.
  - ➢ This can or better say must be observed in the Hodgkin-Huxley model. If this were not possible, the model would not be complete ....
  - ➢ For this we consider the spike pattern in the figure:
    - for t = 20 ms a current pulse trigger an action potential (see first filled arrow).
    - further current pulses at t = 25, 27.5, 30, 32.5 ms are not sufficient to trigger an action potential (see not filled arrows)
    - However at t = 35 ms the same current pulse is able to trigger a second action potential (see second filled arrow)



- The reason for that is not so difficult to get: Since the voltage after an action potential lies far below the zero line, the strength of the current pulse is not sufficient to initiate an action potential. For this, the leakage term R_L is especially important, which leads to a reduction of the potential after the current induction.

3.4 Biological Neuron Models

- From the consideration of the important aspects that can be observed within a spike patterns of a neuron, we still lack another to be complete here. This is adaptation, i.e. the gradual decrease in spike intensity with constant excitation (see upper figure).

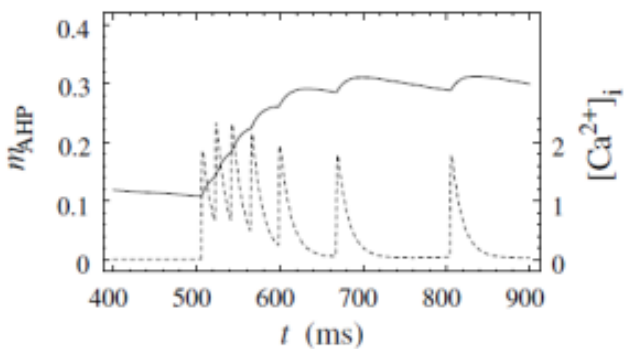


  - Here, in particular, the after hyperpolarization (AHP) is the principal feedback mechanism which control the frequency and patterning of neuronal firing.
  - Such a mechanism can be traced back to a calcium activated potassium current. Which can be taken into account in the model using another ion channel:

$$\sum_k I_K = g_L(u - E_L) + g_{Na} m^3 h (u - E_{Na}) + g_K n^4 (u - E_K) + I_{AHP}$$

with the ion current

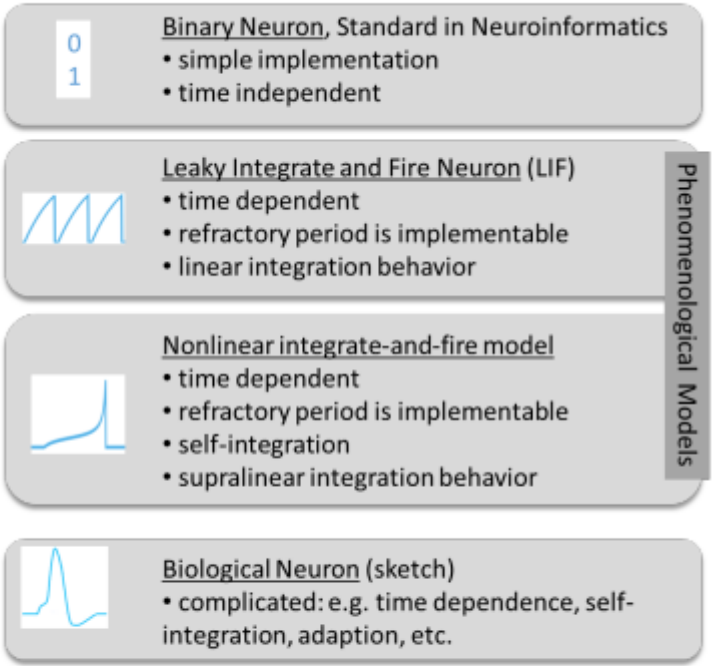
$$\begin{cases} I_{AHP} = g_{AHP}([Ca^{2+}]) m_{AHP} \cdot (u - E_K) \\ \dfrac{dm_{AHP}}{dt} = \alpha m_{AHP} - \beta(1 - m_{AHP}) \end{cases}$$



  - I_{AHP} depends on the calcium concentration within the synaptic gap, which, as shown in the figure (dashed line), initially makes a strong contribution, but decrease over time course of the spike train.
  - The initially sharp increase in the gating variable m_{AHP}, saturates over time as well. Thus, the contribution I_{AHP} to $\sum_k I_K$ is less over the time course.

3.4 Biological Neuron Models

- Thus with the Hodgkin-Huxley model we have a model with which we can describe the dynamics of the individual ion channels and thus gain insight into the functionality of the bio-chemistry of cells.
- As already mentioned, such a model helps to understand the bio-chemical processes in the cells. However, it is very cumbersome and far too complex to describe networks in which there are a large number of neurons.
- Therefore, there is a desire to make the model more handy: so to go back to a phenomenological model.
- Now the question arises, is that even possible? The answer is yes, and I would like to show this in the following.
- I would essentially like to show that complex biological models can be converted into phenomenological models and how these can then be built up in simple hardware circuits.
- let's go back to the beginning...



3.4 Biological Neuron Models

- We start with the Hodgkin-Huxley model in its full beauty:

$$C\frac{\mathrm{d}u}{\mathrm{d}t} = -\sum_k I_k(t) + I(t) \qquad \sum_k I_k = g_{\mathrm{Na}}\, m^3 h\,(u - E_{\mathrm{Na}}) + g_{\mathrm{K}}\, n^4\,(u - E_{\mathrm{K}}) + g_{\mathrm{L}}\,(u - E_{\mathrm{L}})$$

  - As a first step we like to reduce the variable in so-called concentrated parameters. Therefore we replace *n* and *h* by a single effective variable w:

$(b-h) = an$ and a,b are constants $\Rightarrow w = (b-h) = an,\ h = (b-w),\ \mathrm{n} = \frac{\mathrm{w}}{\mathrm{a}}$ and $m(t) \to m_0\,[u(t)]$

$$C\frac{\mathrm{d}u}{\mathrm{d}t} = \underbrace{-g_{\mathrm{Na}}[m_0(u)]^3\,(b-w)\,(u - E_{\mathrm{Na}}) - g_{\mathrm{K}}\left(\frac{w}{a}\right)^4 (u - E_{\mathrm{K}}) - g_L\,(u - E_L)}_{F(u,w)} + I$$

$$R = g_L^{-1},\ \tau = RC \qquad \boxed{\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{\tau}\left[F(u,w) + RI\right] \qquad \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{1}{\tau_w} G(u,w)}$$

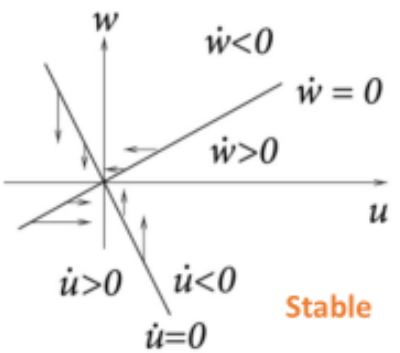
- This gives us the familiar form of the general LIF neuron model again. Only in the difference that we now have 2 variables *w* and *u*. The time dynamics of *w* can be summarized by a second differential equation and thus obtained in the model with concentrated parameters.

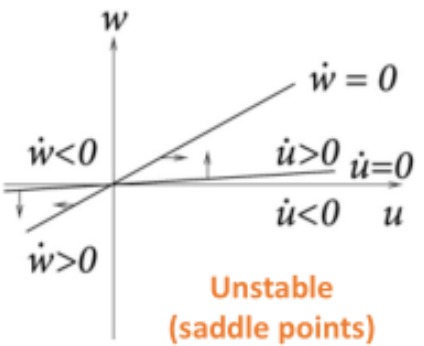
3.4 Biological Neuron Models

- In order to understand this rather general form for describing a neuron a little better and to see what generality is behind this representation, we consider the example of the FitzHugh-Nagumo neuron model:
  - FitzHugh and Nagumo claimed that for a discussion of action potential generation, the four equations of Hodgkin and Huxley can be replaced by two equations.
  - They obtained sharp pulse-like oscillations reminiscent of trains of action potentials by defining the two functions:

$$F(u,w) = u - \frac{1}{3}u^3 - w \quad \text{and} \quad G(u,w) = b_0 + b_1 u - w$$

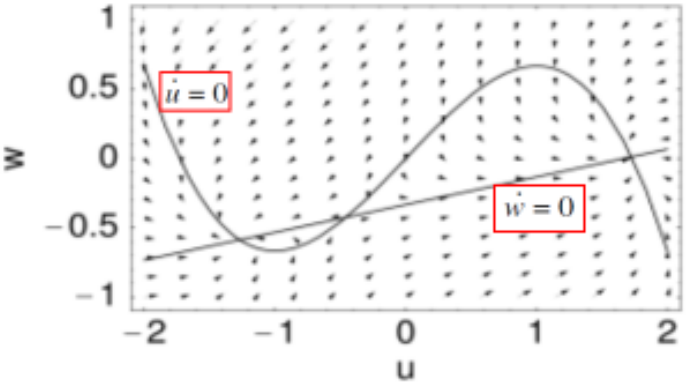
  - A relatively elegant form of analysis of such two-dimensional neuron models can be made using a phase plane analysis, which is explained graphically in the pictures below:
  - The illustration shows 2 phases portraits around a fixed point:
    - The u-nullcline is the set of points were $\dot{u} = 0$
    - The w-nullcline is the set of points were $\dot{w} = 0$
    - the fixed points of the system, defined by $\dot{u} = \dot{w} = 0$
    - The arrows in the phase portrait are vertical on the u-nullcline and arrows are horizontal on the w-nullcline

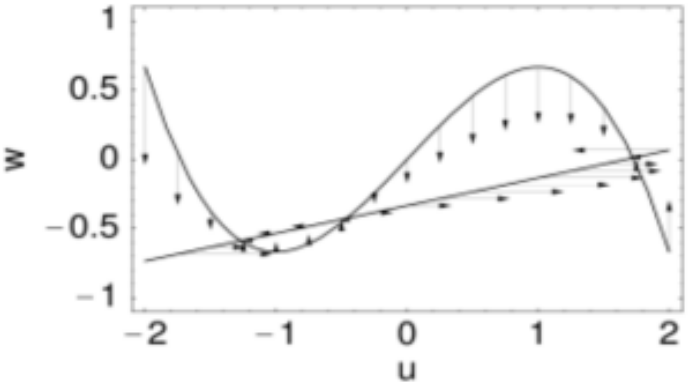




3.4 Biological Neuron Models

- Let's apply this procedure to the FitzHugh-Nagumo neuron model, i.e. we sign the phases portraits :
  - The u-nullcline (curved line) and the w-nullcline (straight line) intersect at the three fixed points.
  - The direction of the arrows indicates the flow.
  - Arrows on the u-nullcline point vertically upward or downward, while on the w-nullcline arrows are horizontal.
  - In the neighbourhood of the fixed points arrows have a short length indicating slow movement.
  - At the fixed point, the direction of the arrows change.

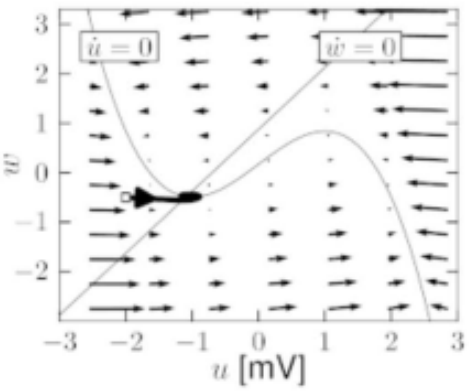


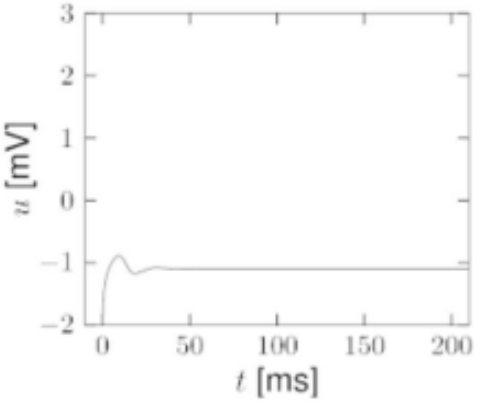


- next we want to look at what happens to the membrane voltage under external excitation, i.e. with the induction of a current value.

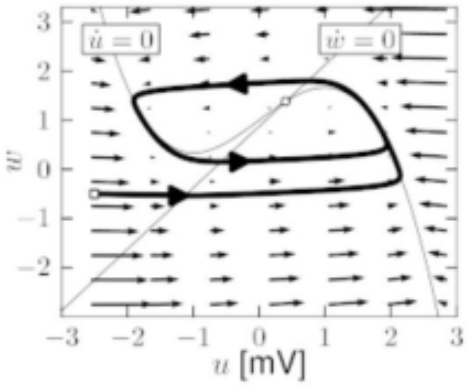
3.4 Biological Neuron Models

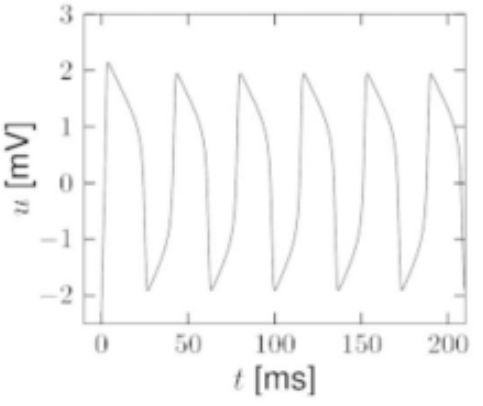
- But, first let's see what happens to the mebrane voltage *u* for zero input:
  - In the figures on the right we see that the phase portrait shows a fixed point without external stimulation.
  - This means that there is no oscillation, so spiking of the neuron is not possible.





- Now, let's give a positive input:
  - In the figures on the right we see that the fixed point is replaced by a so-called limit cycle (see bold line)
  - This enables constant oscillation, as can be seen in the right of the two diagrams.





3.4 Biological Neuron Models

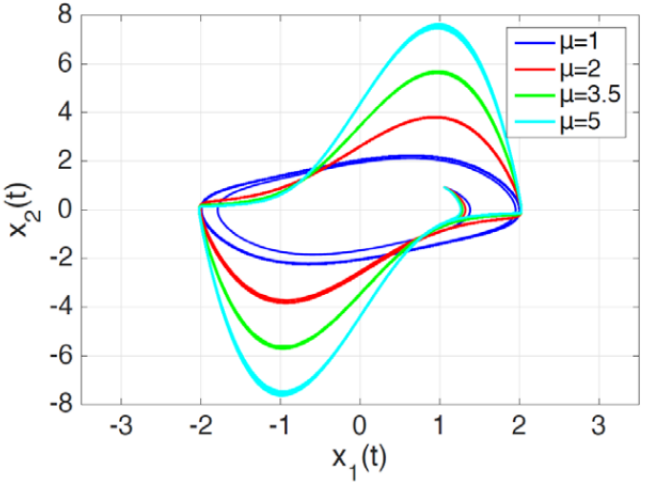
- Now we are so good at it and can further increase the level of abstraction of neuron models…
  - In the last example we saw that neurons can be described within an oscillator model. A very general oscillator model was set up by the Dutch physicist Balthasar van der Pol in 1927, who set up the following universal equation, which describes a large number of oscillators:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

  - The equation describes an oscillatory system with non-linear damping and self-excitation.
  - For small amplitudes the damping is negative (the amplitude is increased);
  - From a certain threshold value of the amplitude, the damping becomes positive, the system stabilizes and goes into a limit cycle.
  - The Liénard transformation converts the 2nd order differential equation into a 1st order one:

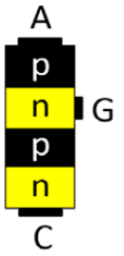
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \mu(1 - {x_1}^2)x_2 - \omega^2 x_1 \end{bmatrix}$$

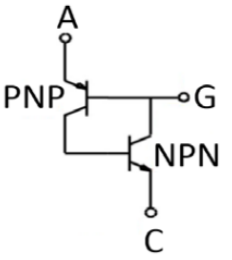
  - The phase portrait of the van der pol oscillator is shown in the figure.

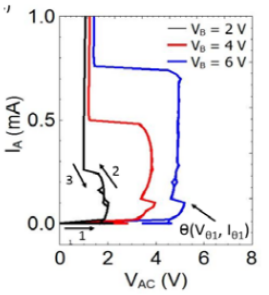


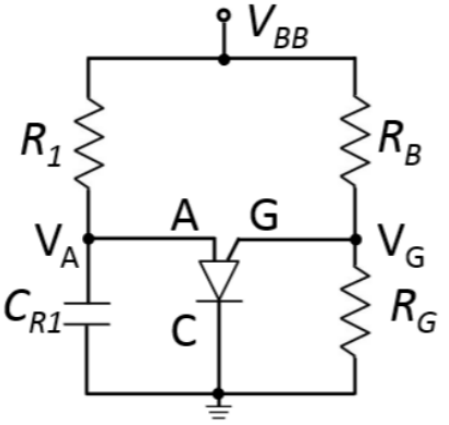
3.4 Biological Neuron Models

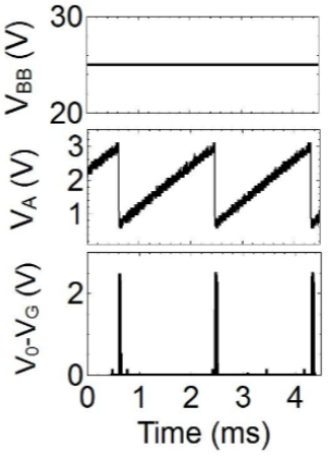
- Finally, I would like to show you that such a van der Pol oscillator can be easily implemented as an electronic circuit. Hence, we are back at neuromorphic circuits…
  - Therefore, we can a programmable unijunction transistor (PUT), which owns a negative differential resistance (see the I-V curve) and consist of a *pnpn* structure, as shown on the figure on the right side, and is a four-layered thyristor.
  - The self-sustained oscillator circuit (see figure below) consists of a resistor-capacitor integrator circuit at the anode (A) of the PUT.
  - A voltage divider consisting of two linear resistors (R_B, R_G) comprises the gate side (G) of the PUT.
  - A constant voltage V_{BB} applied to the oscillator circuit results in a charging of the capacitor.
  - If the capacitor voltage $V_C = V_A$ reaches the threshold V_θ of the PUT, it switches from off to on which leads to a sudden capacitor discharge: oscillation at the anode A can observed (see figure)











3.4 Biological Neuron Models

- I would like to end the chapter of the biologically inspired neuron models with a very brief summary.
  - ➢ We could see that with the rather complex Hodgkin and Huxley neuron model consisting of 4 equations, can be described well the bio-chemical processes of the ion channels.
  - ➢ The complexity of this model could be significantly reduced by the model of FitzHugh and Nagumo and still contains essential components of the Hodgkin and Huxley neuron model, although it only has 2 equations and not 4 equations.
  - ➢ Finally we have a neuromorphic circuit for which we have previously expanded the model FitzHugh and Nagumo to a relaxation oscillator described by the van der Pol equation.
- In the following chapter, we would like to continue here and ask ourselves the question of how can neurons be implemented as effectively as possible in hardware?
  - This especially requires good technology and the best we have at the moment is silicon technology ....but let's see this in the next chapter