Exercise sheet 4
(3.11.22)

1. Harmonic Oscillator

The Harmonic Oscillator has the Hamiltonian corresponding to a particle with a mass \( m \) in the potential

\[
V(x) = \frac{1}{2}m\omega^2x^2,
\]

and the corresponding Schrödinger equation is

\[
\left[ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2 \right] \psi(x) = E\psi(x)
\]

The eigenstates are given by (for \( n = 1, 2, ... \))

\[
\psi_n(x) = A_n(\hat{a}_+)^n\psi_0(x)
\]

with the energies

\[
E_n = \left( n + \frac{1}{2} \right) \hbar\omega.
\]

Where \( \hat{a}_+ \) is so-called raising operator

\[
\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}} (-i\hat{p} + m\omega x) = \frac{1}{\sqrt{2\hbar m\omega}} \left( -\hbar \frac{\partial}{\partial x} + m\omega x \right),
\]

the ground state (\( n = 0, E_0 = \hbar\omega/2 \)) wave function is

\[
\psi_0(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{\hbar}\frac{x^2}{2}},
\]

and \( A_n \) are the normalization constants.

a) Construct \( \psi_1(x) \) and \( \psi_2(x) \).

b) Sketch \( \psi_0(x), \psi_1(x) \) and \( \psi_2(x) \).

c) Check the orthogonality of \( \psi_0(x), \psi_1(x) \) and \( \psi_2(x) \) by explicit integration.

   Hint: If you use the properties of odd and even functions, you only need to do one integral.

d) Calculate \( \langle x \rangle, \langle x^2 \rangle, \langle p \rangle, \langle p^2 \rangle \) for the ground state and the first excited state by explicit integration.

   Hint: introduce the variable \( \xi = \sqrt{m\omega/\hbar}x \) and the constant \( \alpha = (m\omega/\pi\hbar)^{1/4} \).

e) Calculate the expectation value of the kinetic energy and the potential energy (no new integral needed). Is their sum what you would expect?
2. **Angular Momentum**

Classically, the angular momentum of a particle (with respect to the origin) is given by the formula

\[ L = r \times p, \]

or

\[ L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x. \]

The corresponding quantum operators is \( \hat{L} = r \times \hat{p}. \)

a) Starting with the canonical commutation relations for position and momentum \([\hat{x}, \hat{p}] = \hat{x} \hat{p} - \hat{p} \hat{x} = i\hbar\), work out the following commutators:

\[ [\hat{L}_z, x] = i\hbar y, \quad [\hat{L}_z, y] = -i\hbar x, \quad [\hat{L}_z, z] = 0, \]

\[ [\hat{L}_z, \hat{p}_x] = i\hbar \hat{p}_y, \quad [\hat{L}_z, \hat{p}_y] = -i\hbar \hat{p}_x, \quad [\hat{L}_z, \hat{z}] = 0. \]

b) Use these results to obtain \([\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y.\)

c) Find the commutators \([\hat{L}_z, r^2]\) and \([\hat{L}_z, \hat{p}^2]\), where \( r^2 = x^2 + y^2 + z^2 \) and \( \hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2.\)

d) Show that \( \hat{H} = (\hat{p}^2/2m) + V(r) \) commutes with all three components of \( \hat{L} \), provided that \( V \) depends only on \( r \). Thus \( \hat{H}, \hat{L}^2 \) and \( \hat{L}_z \) are mutually compatible observables.