Technische Universität Chemnitz, Fakultät für Mathematik
Christian Sevenheck, Peter Spacek

Sheet 0
on 14.10.2021
due 19.10.2021

Exercises Algebraic Geometry

1. Draw pictures of the following zero sets $V(f)$ in $\mathbb{R}^n$ for the following polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$:
   (a) $f = y - (x^2 - 8x + 15) \in \mathbb{R}[x, y]$;
   (b) $f = y - (x^3 + 3x^2 - 4x) \in \mathbb{R}[x, y]$;
   (c) $f = z - (x^2 + y^2) \in \mathbb{R}[x, y, z]$;
   (d) $f = y^2 - 4x^2 \in \mathbb{R}[x, y]$;
   (e) $f = y^2 + 2xy - 3x^2 \in \mathbb{R}[x, y]$;
   (f) $f = -xz^2 - y^2z + yz^2 - x^2y + x^2z + xy^2 \in \mathbb{R}[x, y, z]$.
   (Hint: What happens when you substitute $x = y$? Or try adding $xyz - xyz$ and factorizing.)

2. Let R be a (commutative[1]) ring, and write $R_n = R[x_1, \ldots, x_n]$. For $f \in R_n$, write $\langle f \rangle \subset R$ for the ideal generated by f, i.e. $\langle f \rangle = \{f \cdot g \mid g \in R_n\}$. For $S \subset R_n$ a subset, write $\langle S \rangle$ for the ideal generated by S, i.e. $\langle S \rangle = \{f_1 \cdot g_1 + f_2 \cdot g_2 + \ldots \mid f_i \in S,\ g_i \in R\}$.
   (a) Write down five examples of elements of $\langle x^2 - 3 \rangle \in R[x]$.
   (b) Is $x^2 + y^2$ an element of the ideal $\langle x^2 + y, x + y^2 \rangle$?
   (c) Is xy an element of the ideal $\langle xy + y^2 + y, x + y \rangle$?
   (Bonus: give the two generators for the ideal of minimal degree.)

3. Let k be a field and R a ring.
   (a) State the following definitions:
      i. an *(integral) domain* R;
      ii. a *unique factorization* domain (UFD) R;
      iii. a *principal ideal* domain (PID) R.
   (b) Give a short argument why: R is a PID $\Rightarrow$ R is a UFD $\Rightarrow$ R is integral.
   (c) Prove that $R = k[x, y]$ is a UFD.
   (d) Argue that $R = k[x, y]$ in *not* a PID.

   In fact, $R = k[x, y]$ is both a *k-module* and a *k-algebra*.

   (e) What is the difference between a k-module and a k-algebra?
   (f) When is a k-module finitely generated?
   (g) When is a k-algebra finitely generated?
   (h) Argue that $k[x, y]/\langle y^2 - x \rangle$ is a finitely-generated $k[x]$-module.

   (Bonus: Is $k[x, y]/\langle xy \rangle$ a finitely-generated $k[x]$-module?)

[1] In this course, rings will always be assumed to be commutative.