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Sheet 1
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Exercises Algebraic Geometry

1. Let k be a field, $I_1, I_2 \subset k[x_1, \dots, x_n]$ ideals and X_1, X_2 affine-algebraic sets in k^n.
   (a) Show that $I_1 \subset I_2$ implies $V(I_1) \supset V(I_2)$.
   (b) Suppose $X_1 \subset X_2$. Show that $I(X_2) \subset I(X_1)$.
   (c) Show that $V(I_1 \cap I_2) = V(I_1 \cdot I_2) = V(I_1) \cup V(I_2)$. Give an example where $I_1 \cdot I_2 \subsetneqq I_1 \cap I_2$.
   (d) Show that $V(I_1 + I_2) = V(I_1) \cap V(I_2)$.

2. Let R be Noetherian and A be a finitely generated R-algebra. Show that A is also Noetherian. (Hint: Study inclusion of ideals in factor rings.)

3. Let k be a field. Find for the following k-algebras R a maximal system of algebraically independent elements $b_1, \dots, b_r \in R$ such that $k[b_1, \dots, b_r] \subset R$ is an integral extension:
   (a) $R = k[x, y]/\langle xy \rangle$;
   (b) $R = k[x, y, z]/\langle x^2 + y^2 + z^2 \rangle$;
   (c) $R = k[w, x, y, z]/\langle xw - yz, zx - y^2 \rangle$;
   (d) $R = k[x, y]/\langle y^2 - x^2 - x^3 \rangle$;
   (e) $R = k[t^3, t^4]$.

   (Hint: follow the proof of Noether's Normalization Theorem.)

4. Draw pictures of the following curves $C = V(f)$ in an appropriate neighborhood of the origin. You may use the program *GeoGebra* (available under `https://www.geogebra.org/`) to do so, but you should at least try to "figure out" how to obtain these pictures (or better draw them by hand)!
   (a) $f = x^2 - y^2 \in \mathbb{R}[x, y]$;
   (b) $f = x^2 + y^2 \in \mathbb{R}[x, y]$;
   (c) $f = y^2 - (x^2 - 1) \cdot (x^2 - 4) \cdot (x^2 - 9) \in \mathbb{R}[x, y]$;
   (d) $f = y^2 - x^3 + x^2 \in \mathbb{R}[x, y]$;
   (e) $f = y^2 - (x^2 - 1) \cdot (x^2 - 4)^2 \in \mathbb{R}[x, y]$.

5. Let K be an algebraically closed field and let $I \subset R_n = K[x_1, \dots, x_n]$ be an ideal. Prove that a point $p = (p_1, \dots, p_n) \in \mathbb{A}^n_K$ lies in $V(I)$ if and only if the ideal $\mathbf{m}_p = \langle x_1 - p_1, \dots, x_n - p_n \rangle$ contains I.