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Sheet 2
on 28.10.2021
due 02.11.2021

Exercises Algebraic Geometry

1. (Repeated from sheet 1, skip if you have already done it.)
   Let k be a field. Find for the following k-algebras R a maximal system of algebraically independent elements $b_1, \ldots, b_r \in R$ such that $k[b_1, \ldots, b_r] \subset R$ is an integral extension:
   (a) $R = k[x, y]/\langle xy\rangle$;
   (b) $R = k[x, y, z]/\langle x^2 + y^2 + z^2\rangle$;
   (c) $R = k[w, x, y, z]/\langle xw - yz, zx - y^2\rangle$;
   (d) $R = k[x, y]/\langle y^2 - x^2 - x^3\rangle$;
   (e) $R = k[t^3, t^4]$.

   (Hint: follow the proof of Noether's Normalization Theorem.)

2. (a) Let X be an irreducible topological space. Show that $\dim(X) = 0$ if X is Hausdorff.
   (b) Show that the Zariski topology on $\mathbb{A}^n_K$ is not Hausdorff for $n > 0$.
   (c) Show that an open subset $U \subset X$ is dense if and only if for every $U' \subset X$ open and non-empty, we have $U \cap U' \neq \emptyset$.

3. (a) Compute the radical of the following ideals.
   i. $(x^p, y^q) \subset k[x, y]$
   ii. $(x^2y^3, z) \subset k[x, y, z]$
   iii. $(xy, x^2) \subset k[x, y]$
   (b) Show that for any algebraic sets $X_1, X_2 \subset k^n$, we have $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$ and find an example where $I(X_1 \cap X_2) \neq I(X_1) + I(X_2)$. Try to understand the geometric reason for this inequality.

4. Find the decomposition into irreducible components of the following algebraic sets.
   (a) $V(x^n + a_1x^{n-1} + \ldots + a_n) \subset \mathbb{C}$ for any $a_1, \ldots, a_n \in \mathbb{C}$.
   (b) $V(x^2 + y^2 - 1, y^2 - x^3 - 1) \subset \mathbb{C}^2$
   (c) $V(x^2 - yz, xz - x) \subset \mathbb{C}^3$