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Sheet 3
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Exercises Algebraic Geometry

1. **The tensor product:** Let A be a ring and M and N two A-modules. We define the *tensor product* $M \otimes_A N$ of M and N over A to be the quotient of the free A-module generated by all symbols $m \otimes n$ with $m \in M$ and $n \in N$ by the submodule generated by all elements of the form

$$\begin{array}{cc} (m_1+m_2)\otimes n - m_1 \otimes n - m_2 \otimes n, & m \otimes (n_1+n_2) - m \otimes n_1 - m \otimes n_2, \\ r(m\otimes n) - m \otimes rn, & rm \otimes n - m \otimes rn, \end{array}$$

for any $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in A$. Show that

(a) The quotient map $\varphi : M \times N \to M \otimes_A N$ given by $(m,n) \mapsto m \otimes n$ is bilinear.[1]

(b) The tensor product has the following universal property:
Any bilinear map $\psi : M \times N \to R$ (with R an A-module) factors uniquely over φ (i.e. there exists a unique A-linear map $\phi : M \otimes_A N \to R$ such that $\psi = \phi \circ \varphi$).

(c) There exist isomorphisms $M \otimes_A A \cong M \cong A \otimes_A M$.

(d) Let $\gamma : M' \to M$ be A-linear, and let P be an A-module, then there is an induced linear map $\gamma \otimes id : M' \otimes_A P \to M \otimes_A P$. (We say that $\cdot \otimes_A P$ is *functorial*.)

A composition of maps

$$M' \xrightarrow{\beta} M \xrightarrow{\gamma} M''$$

is called *exact at* M if $\mathrm{Im}(\beta) = \ker(\gamma)$. A sequence of maps

$$0 \xrightarrow{\alpha} M' \xrightarrow{\beta} M \xrightarrow{\gamma} M'' \xrightarrow{\delta} 0 \tag{1}$$

that is exact at M, M' and M'' is called a *short exact sequence*.

(e) Show that in the short exact sequence (1), $\beta \circ \alpha$, $\gamma \circ \beta$, $\delta \circ \gamma$ are zero; and show that β is injective and γ is surjective

A functor (i.e. a functorial map) F is called *right-exact* if a short exact sequence as in (1) is sent to an exact sequence

$$F(M') \xrightarrow{F(\beta)} F(M) \xrightarrow{F(\gamma)} F(M'') \xrightarrow{F(\delta)} 0.$$

(That is, exact at $F(M)$ and $F(M'')$)

(f) Prove that $\cdot \otimes_A P$ is right-exact. In other words, given the short exact sequence in (1), prove that

$$M'' \otimes_A P \xrightarrow{\beta\otimes\mathrm{id}} M \otimes_A P \xrightarrow{\gamma\otimes\mathrm{id}} M' \otimes_A P \xrightarrow{\delta\otimes\mathrm{id}} 0$$

is exact at $M \otimes_A P$ and $M' \otimes_A P$. Explain that this implies in particular that the map $\gamma \otimes \mathrm{id} : M \otimes_A P \to M' \otimes_A P$ is surjective.

(g) Let $X \subset \mathbb{A}^n_K$ and $Y \subset \mathbb{A}^m_K$ be algebraic sets. Show that the product

$$X \times Y := \{(x,y) \in K^n \times K^m \mid x \in X, y \in Y\} \subset K^{n+m}$$

is algebraic. Show further that $K[X \times Y] \cong K[X] \otimes_K K[Y]$.
(Hint: Show that $K[x_1, \ldots, x_n, y_1, \ldots, y_m] \cong K[x_1, \ldots, x_n] \otimes_K K[y_1, \ldots, y_m]$.)

[1] Recall that a homomorphism of A-modules is frequently called an *A-linear map*; a map between A-modules is called *bilinear* if it is linear in both its arguments.

2. **Local rings:** By definition, a local ring is a (commutative) ring (with unit) with exactly one maximal ideal. The following exercises are hints to the proof of the following:

   Lemma 1 *A ring R is local if and only if the set $R \setminus R^*$ (i.e. the "non-units") is an ideal in R.*

   (a) Let $x \in R$. Show that x is a unit if and only if $\langle x \rangle = R$.

   (b) Argue using Zorn's Lemma (which implies that any proper ideal in R is contained in a maximal ideal) that
   $$R \setminus R^* = \bigcup_{\mathbf{m} \subset R \text{ maximal}} \mathbf{m}$$
   Argue that this shows "$\Rightarrow$" in the lemma.

   (c) Conversely, suppose that $R \setminus R^*$ is an ideal in R. Show that it is maximal.

   (d) Deduce that there cannot be any other maximal ideal, so that R must be local, thus showing "$\Leftarrow$" of the above lemma.

   To get used to working with local rings, try proving the following statements about them:

   (e) Show that if $(R, \mathbf{m})$ is local, then for any $x \in \mathbf{m}$, the element $1 + x$ is a unit in R.

   (f) Let R be a local ring and $I \subset R$ any ideal. Show that the factor ring R/I is also local.

   (g) Let $\mathbb{R}[[x_1, \ldots, x_n]]$ be the ring of formal power series over $\mathbb{R}$. Show that it is a local ring. Give an explicit expression for the inverse of $1 + x$ for $x \in \mathbf{m}$.

   (h) Show that the polynomial ring $\mathbb{R}[x_1, \ldots, x_n]$ is *not* local.

3. **Weighted grading of polynomials:** Fix n integers $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ and define
   $$k[x_1, \ldots, x_n]_d := \bigoplus_{\substack{i_1, \ldots i_n \\ \sum_j i_j a_j = d}} k\, x_1^{i_1} \cdots x_n^{i_n}.$$

   (a) Show that this defines a grading on the polynomial ring $k[x_1, \ldots, x_n]$.
   Is there a choice of $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ that recovers the standard grading?

   For a fixed $(a_1, \ldots, a_n) \in \mathbb{Z}$, a polynomial in $k[x_1, \ldots, x_n]_d$ is called *quasi-homogenous* (or *weighted homogenous*) of degree d.

   (b) Let $\mathrm{char}(k) = 0$. Show that f is quasi-homogenous of degree d if and only if
   $$\sum_{i=1}^{n} a_i x_i \partial_{x_i} f = d \cdot f.$$