Technische Universität Chemnitz, Fakultät für Mathematik
Christian Sevenheck, Peter Spacek

Sheet 4
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Exercises Algebraic Geometry

1. Let $R = \oplus_{i \geq 0} R_i$ be a graded ring. Show the equivalence of the following statements.
   (a) R is noetherian.
   (b) R_0 is noetherian and $R_+ = \oplus_{i>0} R_i$ is a finitely generated ideal in R.
   (c) R_0 is noetherian and R is a finitely generated R_0-algebra.

2. Let L_1 and L_2 be two disjoint lines in $\mathbb{P}^3$ (over k algebraically closed).
   (a) Show that there exist coordinates such that $L_1 = V_+(X_0, X_1)$ and $L_2 = V_+(X_2, X_3)$.
   (b) Let $Z = L_1 \cup L_2$. Determine the homogeneous radical ideal $\mathfrak{a}$ such that $Z = V_+(\mathfrak{a})$.

3. **Grassmannians.** This exercise gives a first glimpse of a huge part of algebraic geometry, called the theory of moduli spaces. The basic idea is that whenever you want to classify or describe any kind of (geometric) objects, you just gather them together in a set and try to put an extra structure on it, e.g., that of an algebraic variety. This is what is called a moduli space. Here is the most basic example, a variety parametrizing sub-vector spaces of fixed dimension of a given space.

   Let V be an n-dimensional vector space over any field k. Define for any $\ell \leq n$ the *Grassmannian* as
$$\mathrm{Gr}(\ell, V) = \{L \subset V \,|\, L \text{ a linear subspace of } \dim_k(L) = \ell\}.$$
   Choose any basis $e_1, \ldots, e_n$ of V, so $V \cong k^n$, and write $\mathrm{Gr}(\ell, n) = \mathrm{Gr}(\ell, k^n)$.

   (a) Let $M = [m_{ij}] \in \mathrm{Mat}_k(\ell, n)$ be an $\ell \times n$-matrix over k and define the associated subspace $L(M)$ to be the span of the vectors
$$M \cdot \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n m_{1i} e_i \\ \vdots \\ \sum_{i=1}^n m_{\ell i} e_i \end{pmatrix}.$$
   For $I = (i_1, \ldots, i_\ell)$ with $1 \leq i_1 < \ldots < i_\ell \leq n$, define M_I to be the the determinant of the matrix consisting of columns I.
      i. Argue that $\dim_k\big(L(M)\big) = \ell$ if and only if at least one of the M_I is non-zero.
      ii. Show that $L(cM) = L(M)$ for $c \in k^*$.
      iii. Let $L \in \mathrm{Gr}(\ell, n)$, prove that there exists an $M \in \mathrm{Mat}_k(\ell, n)$ such that $L = L(M)$.

   (b) Now, consider the *Plücker embedding*
$$\mathrm{Pl}: \begin{array}{rcl} \mathrm{Gr}(\ell, n) & \longrightarrow & \mathbb{P}^{\binom{n}{\ell}-1} \\ L(M) & \longmapsto & (M_I) \end{array}$$
   Show that this map is well-defined and injective.

   (c) Show that the image $\mathrm{Pl}(\mathrm{Gr}(2, n))$ is a projective subvariety in $\mathbb{P}^{\binom{n}{2}-1}$. Therefore the Plücker embedding endows $\mathrm{Gr}(2, n)$ with the structure of a projective variety.
   (This statement is true for any ℓ but the proof is slightly more involved).

   (d) Show that the projective variety $\mathrm{Pl}(\mathrm{Gr}(2, n))$ has an open affine cover by affine spaces of dimension $2(n-2)$.
   (Once we have discussed the definition of the dimension of projective varieties, we will see that this implies that $\dim \mathrm{Pl}(\mathrm{Gr}(2, n)) = 2(n-2)$.)

   (e) Show that $\mathrm{Gr}(2, 2) = \{\mathrm{pt}\}$, $\mathrm{Gr}(1, 3) \cong \mathrm{Gr}(2, 3) \cong \mathbb{P}^2$ and that $\mathrm{Gr}(2, 4)$ is a quadric in $\mathbb{P}^5$ (i.e., a hypersurface cut out by a quadratic polynomial in $k[x_0, \ldots, x_5]$).