Technische Universität Chemnitz, Fakultät für Mathematik
Christian Sevenheck, Peter Spacek

Sheet 4
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Exercises Algebraic Geometry

1. Recall that a collection of subsets $\mathcal{C}$ of space X forms the closed sets of a *topology* for X if

   - an *arbitrary* intersection of elements of $\mathcal{C}$ is itself an element of $\mathcal{C}$;
   - a *finite* union of elements of $\mathcal{C}$ is an element of $\mathcal{C}$;
   - and the sets $\emptyset$ and X lie in $\mathcal{C}$.

   Let R be a ring, show that $\mathcal{C} = \{V(I) \mid I \subset R \text{ ideal}\}$ forms the closed sets of a topology for $\mathrm{Specm}(R)$ and $\mathrm{Spec}(R)$.
   (Note that $V(I)$ is defined differently for $\mathrm{Specm}(R)$ and $\mathrm{Spec}(R)$; cf. Definition 5.1 and 5.11!)

2. Let R a ring and $\mathfrak{p}_x \in \mathrm{Spec}(R)$. Recall that the *residue class field* at x is defined as the quotient field $\kappa(x) = Q(R/\mathfrak{p}_x)$. Prove the claim in Definition 5.11.5:

   > $\kappa(x)$ is equal to the quotient $R_{\mathfrak{p}_x}/\mathfrak{p}_x R_{\mathfrak{p}_x}$.

   (Here $R_{\mathfrak{p}_x}$ is the localization of R at the prime ideal $\mathfrak{p}_x$.)

3. Prove Lemma 5.12:

   > Let R be a ring, and let $I \subset R$ be an ideal. Then the canonical projection $\pi : R \to R/I$ induces a closed embedding[1] ${}^a\pi : \mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$, with image $V(I) \subset \mathrm{Spec}(R)$. On the other hand, if $S \subset R$ is a multiplicatively closed system, then the canonical map $\iota : R \to S^{-1}R$ induces an open embedding ${}^a\iota : \mathrm{Spec}(S^{-1}R) \to \mathrm{Spec}(R)$, and we have
   > $$\mathrm{Im}({}^a\iota) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \cap S = \emptyset\}$$

4. (a) Describe $\mathrm{Specm}\big(\mathbb{C}\big)$ and $\mathrm{Spec}\big(\mathbb{C}\big)$.
   (b) Describe $\mathrm{Specm}\big(\mathbb{C}[x]\big)$ and $\mathrm{Spec}\big(\mathbb{C}[x]\big)$.
   (c) Describe $\mathrm{Specm}\big(\mathbb{C}[x,y]\big)$ and $\mathrm{Spec}\big(\mathbb{C}[x,y]\big)$.

[1] An *embedding* is an injective continuous map that is homeomorphic onto its image. An embedding is called closed (resp. open) if the image is closed (resp. open).