Exercises Algebraic Geometry

1. **Nakayama's Lemma.** In this exercise we will prove the following result concerning local rings:

   Let $(R, \mathbf{m})$ be a local ring, write $K = R/\mathbf{m}$ for the corresponding field, and let A be an R-module.

   (i.) If A is finitely generated and $A \subset \mathbf{m}A$, then $A = \{0\}$.

   (ii.) If $B, C \subset A$ with B finitely generated and satisfying $B \subset C + \mathbf{m} \cdot B$, then $B \subset C$.

   (iii.) If A is finitely generated, then $\{a_1, \ldots, a_k\}$ form a set of generators for A if and only if the classes $\{[a_1], \ldots, [a_k]\}$ form a set of generators for the K-vector space $A/\mathbf{m}A$. That is:

$$A = \sum_{i=1}^{k} R a_i \quad \Leftrightarrow \quad A/\mathbf{m}A = \sum_{i=1}^{k} K[a_1].$$

   We start by proving statement **(i.)**.

   (a) Argue that every element of $\mathbf{m}A$ can be written as $\sum_{j=1}^{k} m_j a_j$, where $a_j \in A$ are the generators, and where $m_j \in \mathbf{m}$. Conclude from the assumptions that $a_i = \sum_{j=1}^{k} b_{ij} a_j$ for $b_{ij} \in \mathbf{m}$.

   Writing $a_i = \sum_{j=1}^{k} \delta_{ij} a_j$ for δ_{ij} the Kronecker delta (i.e. the entries of the identity matrix Id), we conclude that $\sum_{j=1}^{k} (\delta_{ij} - b_{ij}) a_j = 0$. We can write this as a matrix-vector product as

$$\big(\mathrm{Id} - [b_{ij}]\big) \cdot \mathbf{a} = 0, \tag{1}$$

   where $\mathbf{a}$ is the vector with entries a_j.

   (b) Argue that

$$\det\big(\mathrm{Id} - [b_{ij}]\big) = 1 + b_*,$$

   for some $b_* \in \mathbf{m}$ (you do not need to give this element explicitly).
   Next, show that $1 + b^* \in R$ is invertible.

   Just as for matrices over fields, if a matrix over a ring has an invertible determinant, it allows an inverse matrix. Thus, from (b) we conclude that $\mathrm{Id} - [b_{ij}]$ has an inverse.

   (c) Combine the existence of an inverse with equation (1) to obtain that each $a_j = 0$, and from that conclude that $A = \{0\}$. This proves **(i.)**.

   Continuing with statement **(ii.)**, we now no longer assume that A is finitely generated. However, now have two submodules $B, C \subset A$, with B finitely generated and $B \subset C + \mathbf{m}B$.

   (d) Conclude that $\frac{B+C}{C} \subset \frac{C+\mathbf{m}B}{C}$, and argue that this last module is isomorphic to $\mathbf{m}\frac{B+C}{C}$.

   (e) Use result **(i.)** to conclude that $\frac{B+C}{C} = \{0\}$, i.e. $B + C = C$, which implies **(ii.)**.

   Now we turn to statement **(iii.)**, assuming that A is again finitely generated, and recalling that we wrote K for the field $R/\mathbf{m}$.

   (f) Show the implication $A = \sum_{i=1}^{k} R a_i \;\Rightarrow\; A/\mathbf{m}A = \sum_{i=1}^{k} K[a_1]$.

For the other direction, we want to show that $A \subset \sum_{i=1}^{k} Ra_i$ from $A/\mathbf{m}A = \sum_{i=1}^{k} K[a_1]$.

(g) Argue that statement **(ii.)** implies this statement if we show that $A \subset \left(\sum_{i=1}^{k} Ra_i\right) + \mathbf{m}A$.

(h) Now, let $a \in A$, write $[a] \in A/\mathbf{m}A$, then by the assumption we can write $[a] = \sum_{i=1}^{k} c_i[a_i]$. Argue that this implies that $a - \sum_{i=1}^{k} r_i a_i \in \mathbf{m}A$ where $[r_i] = c_i \in K = R/\mathbf{m}$.

(i) Complete the proof of the statement.

2. **Another description of a sheaf.** Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space X. Let $\mathcal{F}_x$ be the stalk of $\mathcal{F}$ at $x \in X$. Define $\widetilde{\mathcal{F}} := \bigcup_{x \in X} \mathcal{F}_x$ to be the union of all stalks. We have the obvious projection map $\pi : \widetilde{\mathcal{F}} \to X$ which associates to any germ $[f]_x$ in $\mathcal{F}_x$ its base point x. We put a topology on $\widetilde{\mathcal{F}}$ generated by the following open neighborhoods of a given germ $[f]_x$: Let $f \in \mathcal{F}(U)$ be a representative of $[f]_x$ then the set

$$\widetilde{U}_{[f]_x} := \{[f]_y \mid y \in U\}$$

is by definition open in $\widetilde{\mathcal{F}}$.

(a) Verify that π is continuous with respect to this topology.

We call a continuous map $s : U \to \widetilde{\mathcal{F}}$ (where U is open in X) a section (of π or of $\widetilde{\mathcal{F}}$) over U if $\pi \circ s = \mathrm{id}_U$. Denote for any $x \in U$ by $f_x \in \mathcal{F}(U_x)$ a representative for the image (germ) $s(x) \in \mathcal{F}_x$. We denote by $\Gamma(U, \widetilde{\mathcal{F}})$ the set of all continuous sections of $\mathcal{F}$ over U.

(b) Show that any map $s : U \to \widetilde{\mathcal{F}}$ satisfying $\pi \circ s = \mathrm{id}_U$ is a section (i.e. continuous) if and only if for any $y \in U_x$ we have $s(y) = [f_y]_y = [f_x]_y$.

(c) Show that any additional structure of the presheaf descends to $\Gamma(U, \widetilde{\mathcal{F}})$, that is, $\Gamma(U, \widetilde{\mathcal{F}})$ is a ring (resp. k-algebra; resp. module; etc.) if $\mathcal{F}$ is a presheaf of rings (resp. k-algebras; resp. modules; etc.).

Define for any open set $U \subset X$ the following map

$$\phi_U : \begin{array}{c} \mathcal{F}(U) \to \Gamma(U, \widetilde{\mathcal{F}}) \\ f \longmapsto (x \mapsto [f]_x) \end{array}$$

(d) Show that ϕ_U is an isomorphism of abelian groups (rings, ...) if and only if $\mathcal{F}$ is a sheaf (and not only a presheaf).

This shows that for a sheaf, the data $\mathcal{F}(U)$ are equivalent to $\widetilde{\mathcal{F}}$, which is called "espace étalé" or *étale space* of the (pre)sheaf $\mathcal{F}$.

(e) Show that for any presheaf $\mathcal{F}$, the rule $U \mapsto \Gamma(U, \widetilde{\mathcal{F}})$ defines a sheaf—which, by (c), is the same as $\mathcal{F}$ in case $\mathcal{F}$ is itself a sheaf. The sheaf $\widetilde{\mathcal{F}}$ is called the *sheafification* of $\mathcal{F}$.
What is the sheafification of the constant presheaf $\mathcal{F}_X(U) := \mathbb{R}$ on a topological space X?

3. Let R be a ring, let $I \subset R$ be an arbitrary ideal, and denote by $J(R) = \bigcap_{\mathbf{m} \text{ maximal}} \mathbf{m}$ the *Jacobson radical*.

Show that $I \subset J(R)$ implies that $X = \mathrm{Spec}(R)$ is the only open set containing $V(I)$.

In particular, for $R = (R, \mathbf{m})$ a local ring, deduce that X is the only open set containing the (unique) closed point of X.