Exercises Algebraic Geometry

1. Let $f : X \to Y$ be a morphism of irreducible schemes, and let η (resp. θ) be the generic point of X (resp. of Y). Show that f is dominant if and only if $\theta \in f(X)$, and in this case $f(\eta) = \theta$.

2. Let Y be a scheme, and let $Z \subset Y$ be a subscheme and denote the inclusion map by i. Then a morphism $f : X \to Y$ of schemes factors through the subscheme Z if and only if the following conditions are satisfied:
   (1) $f(X) \subset Z$ (set-theoretically),
   (2) $f^\flat : \mathcal{O}_Y \to f_*\mathcal{O}_X$ factors through the surjective homomorphism $\mathcal{O}_Y \to i_*\mathcal{O}_Z$.

   Prove that (1) implies (2) if Z is an open subscheme, or if X is reduced.

3. Let X be a noetherian scheme. Show that the nilradical $\mathcal{N}_X$ is nilpotent (i.e., there exists an integer $k \geq 1$ such that $\Gamma(U, \mathcal{N}_X)^k = 0$ for every open subset $U \subset X$).

4. Let X be a scheme.

   (a) If X is affine, show that X_{red} is affine.

   (b) Assume that X is noetherian. If X_{red} is affine, show that X is affine.
   Hint: Use that $\mathcal{N}_X$ is nilpotent (Exercise 3) and reduce to the case $\mathcal{N}_X^2 = 0$. Then show that the canonical morphism $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ is an isomorphism.