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Sheet 8
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Exercises Algebraic Geometry

1. Let k be a perfect field. Decide which of the following rings are regular:
   (a) $R_1 = k[x,y]/\langle 1+x^2+x^2\rangle$
   (b) $R_2 = k[x,y]$
   (c) $R_3 = k$
   (d) $R_4 = \mathbb{Z}[\sqrt{-5}]$
   (e) $R_5 = k[t]/\langle t^n\rangle$

2. Let $R = k[x,y]/\langle x^2-y^3\rangle$ with k a field.
   (a) Determine the generators and relations for $\Omega_{R/k}$.

   Recall that an element $m \in M$ of a module over a ring R is called a *torsion element* if there is a *regular element* $r \in R$ (i.e. r is *not* a zero-divisor) such that $r \cdot m = 0$, in this case r is called an *annihilator* of m. The submodule $\mathrm{tor}(M) = \{m \in M \mid \exists r \in R : r \cdot m = 0\}$ is called the *torsion submodule* of M. Given a submodule $T \subset M$ (e.g. $T = \mathrm{tor}(M)$), the *annihilator* of T is the ideal $\mathrm{ann}(T) = \{r \in R \mid \forall m \in T,\ r \cdot m = 0\}$.

   (b) Calculate the torsion submodule $\mathrm{tor}(\Omega_{R/k})$ for the given R. What is the annihilator of $\mathrm{tor}(\Omega_{R/k})$?
   (c) [Bonus.] For *commutative* rings R, the *support* of a module over R can be defined as $V(\mathrm{ann}(M))$, where $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supset I\}$. What is the support of $\mathrm{tor}(\Omega_{R/k})$? (The usual definition of the support is $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$, see also Exercise 3.)
   (d) [Extra.] Repeat the exercise with $R = k[x,y]/\langle x^2-y^2\rangle$.

3. The *de Rham complex.* Recall that the second exterior power of an R-module M, denoted $\bigwedge^2 M$ or also $\Lambda^2 M$, is obtained from the second tensor power $\bigotimes^2 M = M \otimes M$ by taking the quotient with respect to the ideal I generated by $m_1 \otimes m_2 - m_2 \otimes m_1$. In other words, $\bigwedge^2 M$ is the R-module spanned by elements of the form $m_1 \wedge m_2$ with relations $m_2 \wedge m_1 = -m_1 \wedge m_2$ and $r(m_1 \wedge m_2) = (rm_1) \wedge m_2 = m_1 \wedge (rm_2)$. Higher powers are defined similarly, or using $\bigwedge^k M = M \wedge \left(\bigwedge^{k-1} M\right)$ in combination with the associativity of the tensor product. See also Appendix A2.3 of Eisenbud's *Commutative Algebra with a View towards Algebraic Geometry* for further properties.

   Let S be an algebra over a ring R. Write $\Omega^0_{S/R} = S$ and $\Omega^1_{S/R} = \Omega_{S/R}$, and define $\Omega^k_{S/R} = \bigwedge^k \Omega_{S/R}$. Let $\mathrm{d}^k : \Omega^k_{S/R} \to \Omega^{k+1}_{S/R}$ be defined by the equation

   $$\mathrm{d}^k(s_0\,\mathrm{d}s_1 \wedge \ldots \wedge \mathrm{d}s_k) = \mathrm{d}s_0 \wedge \mathrm{d}s_1 \wedge \ldots \wedge \mathrm{d}s_k,$$

   where $\mathrm{d} : S \to \Omega_{S/R}$ is the universal R-derivation. (Note that $\mathrm{d}^0 = \mathrm{d}$.)

   (a) Show that d^k is well-defined and linear, and that we obtain a *complex*

   $$0 \longrightarrow \Omega^0_{S/R} \xrightarrow{\mathrm{d}^0} \Omega^1_{S/R} \xrightarrow{\mathrm{d}^1} \Omega^2_{S/R} \to \ldots$$

   called the *de Rham complex* of S (relative to R); in other words, show that $\mathrm{d}^{k+1}\mathrm{d}^k = 0$ for all $k \geq 0$.

An analogue of the de Rham complex as defined above is used in differential geometry to compute the cohomology of manifolds. The complex as defined here in the setting of algebraic geometry has many properties in common; for example, if $R = \mathbb{C}$ and S is the affine ring of smooth affine variety X, then the homology of this complex coincides with the topological (e.g. singular) homology of X as a topological space.

The following questions illustrate other parallels between the settings of algebraic and differential geometry:

Smoothness of an affine variety over a field. Let M be an S-module, and let $\mathfrak{p} \in \operatorname{Spec}(S)$ be a prime ideal. We can define a $S_\mathfrak{p}$-module $M_\mathfrak{p}$ by allowing division of elements in M by the elements of $S \setminus \mathfrak{p}$, i.e. $M_\mathfrak{p} = S_\mathfrak{p} \otimes_S M$. We say that M satisfies a property *locally* if for all $\mathfrak{p} \in \operatorname{Spec}(S)$ the $S_\mathfrak{p}$-module $M_\mathfrak{p}$ satisfies the property. In particular, we call M *locally free* if $M_\mathfrak{p}$ is a free $S_\mathfrak{p}$-module for every $\mathfrak{p} \in \operatorname{Spec}(S)$.

Now, let $X = \operatorname{Spec}(S)$ be an affine variety of dimension d over a *field* R. The variety X is called *smooth* (over R) if $\Omega_{S/R}$ is locally free of rank d as an S-module.

(b) Show that $\Omega^k_{S/R} = 0$ for all $k > d$ if $X = \operatorname{Spec}(S)$ is a smooth affine variety of dimension d.

(This corresponds to the fact that (complex/real) manifolds of dimension d have trivial (complex/real) cohomology groups in degrees higher that d.)

The cohomology of the plane and the pointed plane. Let R be ring. Recall that the affine plane of dimension n is defined as $\mathbb{A}^n_R = \operatorname{Spec}(R_n)$ for $R_n = R[x_1, \dots, x_n]$. Also recall that $\mathbb{A}^1_R \setminus \{0\}$ is the spectrum of $R[x, x^{-1}] = R[x, y]/\langle xy - 1\rangle$.

(c) Show that the de Rham complex of R_n is exact at $\Omega^k_{R_n/R}$ for all $k \neq 0$ (i.e. $\ker(\mathrm{d}^k) = \operatorname{Im}(\mathrm{d}^{k-1})$ for all $k \neq 0$).

(This corresponds to the fact that the cohomology of the plane as a manifold vanishes for non-zero degrees.)

(d) Let R be a field. Show that $\Omega_{S/R}$ is free of rank 1 for $S = R[x, x^{-1}]$ with generator $\mathrm{d}x$. Conclude that the de Rham complex becomes

$$0 \to S \xrightarrow{\mathrm{d}} \Omega_{S/R} \to 0,$$

with zeroth homology (i.e. $\ker \mathrm{d}$) equal to R and first homology (i.e. $\Omega_{S/R}/\operatorname{Im}\mathrm{d}$) equal to R as well.

(This corresponds to the fact that the pointed plane is homotopy equivalent to the circle.)