Exercises Algebraic Geometry

1. Let k be an algebraically closed field with $\operatorname{char}(k) = 0$ (so in particular, k is perfect).
   (a) Consider the variety $X_t = \operatorname{Spec}(R_t)$ for $R_t = k[x,y]/\langle f_t\rangle$ for $f_t(x,y) = x^2 - y^3 + ty$. Recall that for $t = 0$, X_0 is singular. Is X_t singular for other values of $t \in k$?
   (b) Now, let $F(x,y,t) = f_t(x,y) \in k[x,y,t]$ and let $Y = \operatorname{Spec}(k[x,y,t]/\langle F\rangle)$. Is Y smooth? If not, what is the singular locus?
   (c) Determine the singular loci of the following spectra:
      i. $\operatorname{Spec}\big(k[x,y]/\langle (x-y)^2\rangle\big)$.
      ii. $\operatorname{Spec}\big(k[x,y]/\langle xy(1-x-y)\rangle\big)$.
      iii. $\operatorname{Spec}\big(k[x,y,z]/\langle xy, xz\rangle\big)$.
      iv. $\operatorname{Spec}\big(k[x,y,z]/\langle x(z-x^2-y^2), y(z-x^2-y^2)\rangle\big)$.

For exercises 2 and 3, let $(X, \mathcal{O}_X)$ be a locally ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module. The *fiber* of $\mathcal{F}$ in $x \in X$ is defined as the $\kappa(x)$-vector space

$$\mathcal{F}(x) = \mathcal{F}_x/\mathbf{m}_x\mathcal{F}_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x),$$

where $\kappa(x) = \mathcal{O}_{X,x}/\mathbf{m}_x$ is the residue field at x. For $s_x \in \mathcal{F}_x$ a germ, we denote by $s(x)$ its image in the fiber.

2. Let $\mathcal{F}$ be of finite type.
   (a) Show that $\operatorname{supp}(\mathcal{F}) = \{x \in X \mid \mathcal{F}(x) \neq 0\}$.
   (b) If $\mathcal{G}$ is another $\mathcal{O}_X$-module of finite type, show that
   $$\operatorname{supp}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) = \operatorname{supp}(\mathcal{F}) \cap \operatorname{supp}(\mathcal{G})$$
   (c) Let $f : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$ be a morphism of locally ringed spaces. Show that
   $$\operatorname{supp}\big(f^*(\mathcal{F})\big) = f^{-1}\big(\operatorname{supp}(\mathcal{F})\big).$$

3. Let $\mathcal{F}$ be of finite type and let $s_1, \dots, s_r \in \Gamma(X, \mathcal{F})$.
   (a) Show that $\{x \in X \mid s_1(x), \dots, s_r(x) \text{ generate } \mathcal{F}(x)\}$ is open in X.
   (b) Show that $\{x \in X \mid s_1(x), \dots, s_r(x) \text{ are linearly independent in } \mathcal{F}(x)\}$ is open in X if $\mathcal{F}$ is a finite locally free $\mathcal{O}_X$-module.