18.10.21

Algebraic geometry

Plan: affine algebraic varieties

- Hilbert basis theorem
- State Hilberts Nullstellensatz

Notation: K field, $n \in \mathbb{N} = \mathbb{N}_0$

$R_n := K[X_1, \dots, X_n]$; $\mathbb{A}^n_K := K^n$

Definition 2.1.: K field, $S \subset R_n$

Then we call

$$V(S) := \{ (x_1, \dots, x_n) \in \mathbb{A}^n_K \mid P(x_1, \dots, x_n) = 0 \quad \forall P \in S \}$$

the affine algebraic set defined by S.

<u>examples:</u>

(i) If $A \in (m \times n, K)$, the $\ker(A)$ is affine alg. set, since it is the set of solutions of linear (hence algebraic) equations.

(ii) $f(x) \in R_1 = K[x] \rightsquigarrow$ (finite) set of zeros is affine algebraic

(iii) K alg. closed, let $f(x,y) \in R_2$, $g(x,y,z) \in R_3$ be two non-constant polynomials

$\rightsquigarrow$ $V(\{f\})$ $\qquad$ $V(\{g\})$

affine alg. curve $\qquad$ affine alg. surface

(iv) ex. where $|S| > 1$:

$$S = \{ x \cdot y, x \cdot z, y \cdot z \} \subset K[x, y, z]$$

$$\rightsquigarrow V(S) = \{ (x, 0, 0) \mid x \in K \} \cup \{ (0, y, 0) \mid y \in K \} \cup \{ (0, 0, z) \mid z \in K \} \subset \mathbb{A}^3_K$$

Lemma 2.2.

(a) Let $X \subset \mathbb{A}^n_K$ be an affine algebraic set, then

$$\underbrace{I(X) := \{ P \in R_n \mid P(\underline{x}) = 0 \ \forall \underline{x} \in X \}}_{\subset R_n}$$

is an ideal in R_n, called the vanishing ideal of X in R_n

(b) For any subset $S \subset R_n$, then:

$$V(S) = V(\underbrace{\langle S \rangle}_{:= R_n \cdot S})$$

ideal generated by S

Proof: If for $f, g \in R_n$ we have

$$f_{|X} = g_{|X} = 0 \quad (\text{i.e., } f(x) = g(x) = 0 \ \forall x \in X)$$

$$(\Leftrightarrow f, g \in I(X)), \text{ then } (\forall \lambda \in R_n):$$

$$(\lambda f + g)_{|X} = 0 \quad (\text{hence } \lambda f + g \in I(X))$$

$$S \subset \langle S \rangle \rightsquigarrow V(\langle S \rangle) \subset V(S)$$

If $f_1, \ldots, f_k \in S$, $\lambda_1, \ldots, \lambda_k \in R_n$, $F = \sum \lambda_i f_i$

$$\forall x \in V(S): F(x) = 0$$

$$\Rightarrow V(\langle S \rangle) \subset V(S) \quad \square$$

<u>Question</u>: Can any affine algebraic set be defined by finitely many polynomial equations?

Since it suffices to study $V(I)$, where $I \subset R_n$ ideal

$\rightsquigarrow$ <u>Question</u>: $I \subset R_n$ ideal, $\exists ? \, f_1, \dots, f_k \in R_n : I = (f_1, \dots, f_k)$?

$\Leftrightarrow$ Is R_n noetherian ring?

Theorem 2.3: (Hilbert basis theorem)

Let R be a noetherian ring, then $R[x]$ is also noetherian.

Proof: Suppose $R[x]$ is not noetherian, i.e. $\exists\, I \subset R[x]$ that cannot be finitely generated. This means that $\exists$ sequence $f_1, f_2, f_3, \dots \in I$ s.t. $f_{i+1} \notin (f_1, \dots, f_i)$ and s.t. f_{i+1} has minimal degree among all such elements.

Write

$$f_i(x) = a_i \cdot X^{\deg(f_i)} + \text{(Terms of lower order)}$$

Put $J_i := (a_1, \dots, a_i) \subset R$

then by definition $J_1 \subset J_2 \subset \dots$

Since R noetherian, this chain of ideals in R becomes stationary, i.e.

$\exists\, r \in \mathbb{N}: \; a_r \in (a_1, \dots, a_{r-1})$, hence

$$a_r = \sum_{i=1}^{r-1} \lambda_i \cdot a_i \qquad \lambda_i \in R$$

Put: $f :\overset{(*)}{=} \underset{\substack{\| \\ a_r \cdot x^{\deg(f_r)}}}{f_r} - \sum_{i=1}^{r-1} X^{\deg(f_r) - \deg(f_i)} \lambda_i \underset{\substack{\| \\ a_i x^{\deg(f_i)} + (\text{lot})}}{f_i}$

Then: - $f \in I$

- $f \notin (f_1, \ldots, f_{r-1})$: If we had

$f \in (f_1, \ldots, f_{r-1}) \Rightarrow (*)$ would imply

$f_r \in (f_1, \ldots, f_{r-1})$ ↯

degree of f:

$$f = a_r \cdot X^{\deg(f_r)} + (\text{l.o.t.}) - \sum_{i=1}^{r-1} \lambda_i a_i \cdot X^{\deg(f_r)} + (\text{lot})$$

$$= \left(a_r \cdot X^{\deg(f_r)} - \sum \lambda_i a_i \, X^{\deg(f_r)}\right) + (\text{lot})$$

$$= \underbrace{\left(a_r - \sum_{i=1}^{r-1} \lambda_i a_i\right)}_{=0} x^{\deg(f_r)} \notin (Lo\,f)$$

$$= (Lo\,f)$$

$$\Rightarrow \deg(f) < \deg(f_r) \quad \lightning$$

$\square$

<u>Corollary 2.4:</u> Let K be a field

(i) $R_1 = K[x]$ noetherian

(ii) R_n is noetherian (by induction)

Summary:

$\exists$ correspondences:

$$\left\{\begin{array}{c}\text{affine alg.}\\ \text{sets in } \mathbb{A}^n_k\end{array}\right\} \underset{V(-)}{\overset{I(-)}{\rightleftarrows}} \left\{\begin{array}{c}\text{ideals}\\ \text{in } R_n\end{array}\right\}$$

<u>Q:</u> Are these bijections?

<u>A:</u> NO (in general)

Preparations to turn this into a true statement.

<u>Idea:</u> $f(x) \in R_1 \Rightarrow V(\boxed{(f)}) \overset{\neq}{=} V(\boxed{(f^k)})$

$$V(x) = V(x^2) = \{0\}$$

Def. 2.5.: Let R be a ring

$I \subset R$ an ideal, then

$$\sqrt{I} := \mathrm{rad}(I) := \{ f \in R \mid \exists n \in \mathbb{N}_{>0} : f^n \in I \}$$

is called the radical of I

Rk: (i) $I \subset \sqrt{I}$

(ii) $\sqrt{I} \subset R$ is an ideal (ex:)

The ideal $I \subset R$ is called

radical (or radical ideal)

iff (iff $\hat{=}$ if and only if)

$$\overline{I} = \sqrt{I}$$

(iii) Let $X \subset \mathbb{A}^n_K$ be affine algebraic

$$\Rightarrow \overline{I(X)} = \sqrt{I(X)}$$

(why: "$\supset$": $f \in \sqrt{I(X)}$

$$\rightsquigarrow \exists\, n \in \mathbb{N}_{\geq 1}: f^n \in I(X)$$

$$\Leftrightarrow f^n(x) = 0 \quad \forall x \in X$$

$$\Rightarrow f(x) = 0 \quad \forall x \in X$$

$$\Leftrightarrow f \in I(X)$$)

Lemma 2.6.: Let $I_1, I_2 \subset R_n$ be ideals and let $X_1, X_2 \subset \mathbb{A}^n_K$ be affine alg. sets. Then

1.) $I_1 \subset I_2 \Rightarrow V(I_2) \subset V(I_1)$

2.) Write $I_1 \cdot I_2 := \langle \{ f_1 \cdot f_2 \in R_n \mid f_i \in I_i \} \rangle$ for ideal generated by all products $f_1 \cdot f_2$, then

$$V(I_1 \cdot I_2) = V(I_1 \cap I_2)$$
$$= V(I_1) \cup V(I_2)$$

In particular, the union of 2 affine alg. sets is again affine alg.

3.) $V(I_1 + I_2) = V(I_1) \cap V(I_2)$

hence, the class of affine alg. sets is closed under intersections.

4.) $X_1 \subset X_2 \Rightarrow I(X_2) \subset I(X_1)$

<u>Proof</u>: exercise.

<u>Theorem 2.7.</u> (Hilbert's Nullstellensatz, strong form)

Let K be alg. closed.

Then $\exists$ bijections

$$\left\{\begin{array}{c}\text{affine alg.}\\ \text{varieties in } \mathbb{A}^n_K\end{array}\right\} \underset{V(-)}{\overset{I(-)}{\rightleftarrows}} \left\{\begin{array}{c}\text{radical ideals}\\ \text{in } R_n\end{array}\right\}$$

$I(-)$ and $V(-)$ are inverse to each other and the reverse inclusions.

Pf: to show: o.k.

$$\boxed{\begin{array}{l} - \; V(I(X)) = X \quad \forall X \subset \mathbb{A}^n_K \text{ affine algebraic} \\ - \; I(V(I)) = \sqrt{I} \quad \forall I \subset R_n \end{array}}$$

"$\supseteq$": ok.

- $X \subset V(I(X))$: $\underline{\forall x \in X, \forall f \in I(X)}$
$\underline{\Longrightarrow f(x) = 0}$: $x \in V(I(X))$

- $\sqrt{I} \subset I(V(I))$: $\forall f \in R_n$:
$f^n \in I \rightsquigarrow f^n_{|V(I)} = 0$

$\rightsquigarrow f_{|V(I)} \Rightarrow f \in I(V(I))$

- $V(I(X)) \subset X$:
given $X \subset \mathbb{A}^n_k$: $\exists I$: $X = V(I)$
also: $I \subset \sqrt{I} \subset I(X)$
$\overset{2.6}{\Longrightarrow} V(I(X)) \subset V(I) = X$

- hard part of theorem:

$$\boxed{I(V(I)) \subset \sqrt{I}}$$ to prove

PID

<u>example</u>: $n=1$, $I \subset K[x]$

i.e. $I = (f) \xRightarrow[\text{closed}]{K \text{ alg.}}$

$$f = a_0 \cdot \prod_{i=1}^{k} (x-\lambda_i)^{m_i} \qquad \lambda_i \neq \lambda_j \quad i \neq j$$

$$V(I) = V(f) = \{\lambda_i\}_{i=1}^{k} \subset \mathbb{A}^1_K$$

$$I(V(I)) = (g) \qquad g = a_0 \prod_{i=1}^{k} (x-\lambda_i)$$

$$(g) = \sqrt{(f)}$$ $\square$

Theorem 2.8. (weak form of Nullstellensatz)

Let K be a field, $\overline{K}$ an alg. closure, let $I \subsetneq R_n = K[x_1, \dots, x_n]$ be a proper ideal, then there is some $\underline{a} = (a_1, \dots, a_n) \in \mathbb{A}^n_{\overline{K}} (\hat{=} \overline{K}^n)$ such that $f(a) = f(a_1, \dots, a_n) = 0 \ \forall f \in I$.

Proof of 2.7. using 2.8.:

Proof goes by localization techniques called Rabinowitch's trick:

By assumption: $K = \overline{K}$

to show: $I \subset R_n$

$$I(V(I)) \subset \sqrt{I}$$

$$I = (f_1, \dots, f_k)$$

$$f \in I(V(I)) \overset{?}{\Rightarrow} f \in \sqrt{I}$$

$$J := I + (f \cdot t - 1) \subset K[x_1 \dots x_n, t]$$

<u>Claim</u>: $V(J) = \emptyset$:

$$(x, t) \in V(J) \Rightarrow x \in V(I) \Rightarrow f(x) = 0$$

$$(f \cdot t - 1)(x, t) = -1 \neq 0 \quad \forall (x,t) \in V(J)$$

$$\xRightarrow[K=\overline{K}]{\text{Th. 2.7}} J = K[x_1,\dots,x_n,t] = (1)$$

$\Rightarrow \exists$ coefficients $r_1,\dots,r_k, s \in K[\underline{x},t]$

$$1 = \sum_{i=1}^{r} r_i \cdot f_i + s \cdot (f \cdot t - 1) \quad (**)$$

Consider the ring homomorphism

$$\varphi: K[x_1,\dots,x_n,t] \longrightarrow K(x_1,\dots,x_n) = Q(K[x_1,\dots,x_n])$$

quotient field

$$x_i \longmapsto x_i$$

$$t \longmapsto \frac{1}{f}$$

& $\varphi_{|K} = \mathrm{id}_K$

$\Longrightarrow$

$$\varphi(1) = 1 = \sum_{i=1}^{k} \varphi(r_i) \cdot \varphi(f_i) + \varphi(s) \cdot \left(\varphi(f) f^{-1} - 1\right)$$

$$\varphi_{|\overbrace{K[x_1 \dots x_n]}^{R_n}} = id$$

$$= \sum \varphi(r_i) \cdot f_i + \varphi(s) \left(\underbrace{f \cdot f^{-1} - 1}_{=0}\right)$$

$$= \sum_{i=1}^{k} \underbrace{\varphi(r_i)}_{\frac{p_i}{f^{n_i}}} \cdot f_i$$

for some $p_i \in R_n$

$n_i \in \mathbb{N}$

Take $n := \max(n_i)$, then

$$1 = \frac{1}{f^n} \cdot \sum \underbrace{p_i \cdot f^{\overbrace{n-n_i}^{\geq 0}}}_{\in R_n} \cdot f_i$$

$$\Leftrightarrow f^n = \underbrace{\sum \left(p_i \cdot f^{n-n_i}\right) \cdot f_i}_{\in \overline{I}}$$

$$\Rightarrow f \in \sqrt{\overline{I}}$$

$\square$