Recall:

Theorem 2.8 (Weak Nullstellensatz): K field, $\overline{K}$ alg. closure, $I \subsetneq R_n = K[x_1, \ldots, x_n]$ proper ideal.
Then $\exists\ \underline{a} = (a_1, \ldots, a_n) \in \overline{K}^n$ s.t. $f(\underline{a}) = 0\ \forall f \in I$.

Preparations for proof:

Def. 2.9. (i) Consider rings R, S and a ring homomorphism $R \to S$, then we call S an R-algebra. This means: An R-algebra is a ring that is simultaneously a module over R such that the two structures are compatible.

If $R \to S$ is injective, then we identify R with its image in S.

(ii) An R-algebra S is called finitely generated, iff $\exists$ an isomorphism of R-algebras (i.e. a ring isomorphism which is the identity on R)

$$S \simeq \frac{R[x_1, \ldots, x_n]}{\mathfrak{a}}$$

for some ideal $\mathfrak{a} \subset R[x_1 \ldots x_n]$

In this case, if $s_i \in S$ is the

image in S of the class of x_i, we say that S is generated by $s_1, \ldots, s_n$ and we write $S \cong R[s_1, \ldots, s_n]$.

If S is a finitely generated R-alg., i.e. $S \cong R[s_1, \ldots, s_n]$, then we say that $s_1, \ldots, s_n$ are algebraically independent iff $S \cong R[x_1, \ldots, x_n]$, i.e. the ideal $\mathfrak{a}$ from above is trivial.

<u>Theorem 2.10</u>: Let K be a field, $L \supset K$ a field extension. Suppose that L is a finitely generated K-algebra.

Then $L \supset K$ is a finite,

hence, an algebraic field extension.

Rk: If a field extension $L \supset K$ is finitely generated as K-algebra

$\Longrightarrow$ L is a finitely generated field extension (i.e. $L = K(s_1, \dots, s_n)$)

"$\Longleftarrow$": not true: $\mathbb{Q}(\pi)$ f.g. field extension (of $\mathbb{Q}$), but not algebraic

hence not finite $\overset{2.8.}{\Longrightarrow}$ $\mathbb{Q}(\pi)$ not f.g. as $\mathbb{Q}$-alg.

Proof of 2.8 by 2.10: Consider

alg. closure $\overline{K} \supset K$ and take $I \subsetneq R_n = K[x_1, \dots, x_n] \overset{\text{Zorn}}{\rightsquigarrow} \exists$ maximal ideal

$m \subsetneq R_n$ s.t. $I \subset m$. $\rightsquigarrow \boxed{R_n/m} \supset K$ field extension

1. field
2. f.g. K-alg

$\overset{2.10}{\Longrightarrow}$ $R_n/m \supset K$ finite ($\Rightarrow$ algebraic)

$\overset{\text{theorem}}{\Longrightarrow}$ $\exists \phi : R_n/m \hookrightarrow \overline{K}$

on ext. of field hom.

Take $\underline{a} = (a_1 \dots a_n) := (\phi(\bar{x}_1), \dots, \phi(\bar{x}_n)) \in \overline{K}^n$

Then $\forall f \in I \subset m$, we have

$$f(a_1, \dots, a_n) = 0 \qquad \square$$

---

remains to show 2.10: use theory of ring extensions.

<u>Def. - Lemma 2.11:</u> Consider an A-algebra B, i.e. a ring homomorphism $A \to B$. Let $b \in B$. Then the foll. cond. are equivalent

1.) b is the zero of a unitary polynomial $f \in A[X]$, i.e. $f(b) = 0$

2.) The A-algebra $A[b] \subset B$ (by def., $A[b]$ is f.g. A-alg.) is finitely generated as A-module.

3.) There is a finitely generated A-submodule M of B, i.e.

$\exists\, m_1, \ldots, m_n \in B$ s.t. $M = \sum_{i=1}^{n} A \cdot m_i$

s.t.: - $1 \in M$

- $b \cdot M \subset M$

If b satisfies these conditions, we say that b is integral over A

The k-algebra B is called integral over A iff all $b \in B$ are integral over A.

If $A \subset B$, and if B is integral over A $\Rightarrow$ B is called integral ring extension.

1.) $\Rightarrow$ 2.) We know: $\exists\, a_{n-1}, \dots, a_0 \in A$ s.t.

$$b^n + a_{n-1} \cdot b^{n-1} + \dots + a_1 b + a_0 = 0$$

$$\Leftrightarrow b^n = -a_{n-1} b^{n-1} - \dots - a_0$$

$\Rightarrow$ b^n lies in the A-submodul generated by $b^0, b^1, \dots, b^{n-1}$

Ind

$\leadsto A[b] = \sum_{i=0}^{n-1} A \cdot b^i$

2.) $\Rightarrow$ 3.) clear: $M = A[b]$ $\quad b \cdot A[b] \subset A[b]$

3.) $\Rightarrow$ 1.) Suppose that $\exists$ A-submodul

$M = \sum_{i=1}^{n} A \cdot m_i$ s.t. $\boxed{\begin{array}{l} - 1 \in M \\ - b \cdot M \subset M \end{array}}$

$\leadsto b \cdot m_i \in M \qquad \forall i = 1 \dots n$

$\leadsto \exists$ equations: $b \cdot m_1 = a_{11} \cdot m_1 + \dots + a_{1n} \cdot m_n$

$\vdots$

$b \cdot m_n = a_{n1} \, m_1 + \dots + a_{nn} \cdot m_n$

where $a_{ij} \in A$

$$\rightsquigarrow \boxed{X \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = 0} \quad (*)$$

where $X = \left(\delta_{ij} e - a_{ij} \right) \in M(n \times n, B)$

We have:

$$X^* \cdot X = \det(A) \cdot E_n$$

where X^* is the adjoint matrix

$$\overset{(*)}{\Longrightarrow} 0 = X^* \cdot X \cdot \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \det(X) \cdot E_n \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}$$

$$= \begin{pmatrix} \det(X)\cdot m_1 \\ \vdots \\ \det(X)\cdot m_n \end{pmatrix}$$

$$\Rightarrow \det(X)\cdot m_i = 0 \quad \forall i = 1\dots n$$

$$\xrightarrow{1\in M} \exists \lambda_1 \dots \lambda_n \in A : 1 = \sum_{i=1}^{n} \lambda_i \cdot m_i$$

$$\Rightarrow \det(X) = \det(X)\cdot 1 = \sum_{i=1}^{n} \lambda_i \cdot \underbrace{\det(X)\cdot m_i}_{=0}$$

$$\Rightarrow \det(X) = 0 \qquad X = (\overbrace{\delta_{ij}\cdot b - a_{ij}}^{\in B})_{ij}$$

$$f(x) := \det(\delta_{ij}\cdot x - a_{ij}) \in A[x] \quad b$$

$$f(b) = 0. \qquad \square$$

Rk: If A, B are fields (in part: $A \subset B$) then $b \in B$ integral over $A \iff b$ is algebraic over A.

Lemma 2.12: Let B be an A-alg

a) If B is f.g. as A-module $\implies B$ is integral over A.

b) More generally: If $B = A[b_1, \dots, b_n]$ is only f.g. as A-algebra then if $b_1, \dots, b_n$ are integral over A, so is B.

Proof: a) Lemma 2.11., cond. 3

for $M = B$

b) by induction over n

$$A \subset A[b_1] \subset A[b_1, b_2] \subset \dots \subset A[b_1, \dots, b_n] = B$$

By 2.11., 2.) we know that

$\forall i$: $A[b_1, \dots, b_i]$ is a finitely

generated $A[b_1, \dots, b_{i-1}]$

$\dots\dots$ B is f.g. A-module.

$\overset{a)}{\Longrightarrow}$ B integral over A.

Lemma 2.13: Let $A \subset B$ be an integral ring extension and s.t. A, B are integral domains (i.e. they have no zero-divisors)

Then: A field $\iff$ B field.

Pf: "$\Rightarrow$": Suppose that A is a field, take $b \in B \setminus \{0\}$ $\leadsto$ b integral over A, i.e. $\exists\, a_{n-1}, \dots, a_0 \in A$:

$$b^n + a_{n-1} b^{n-1} + \dots + a_0 \overset{(*)}{=} 0$$

$\{\frac{a}{b} \mid b \neq 0\}$, $a, b \in B$

B integral domain $\leadsto$ consider $Q(B)$

quotient field, then (*) holds in $Q(B)$ as well, assume $a_0 \neq 0$ (if not, divide by sufficiently high power of b) $\rightsquigarrow -a_0 = b^n + a_{n-1}b^{n-1} + \dots + ba_1$

$$= b\left(b^{n-1} + a_{n-1}b^{n-1} + \dots + a_1\right)$$

$$\Leftrightarrow 1 = b \cdot \underbrace{\left(-a_0^{-1}\right) \cdot \left(b^{n-1} + a_{n-1}b^{n-1} + \dots + a_1\right)}_{=: b^{-1} \overset{!}{\in} B}$$

(since $-a_0^{-1} \in A$, since A field)

$\rightsquigarrow B$ is field

"$\Leftarrow$": Suppose: B field

$a \in A \setminus \{0\}$: $\exists\, a^{-1} \in B$

$\rightsquigarrow a^{-1}$ satisfies equation:

$$a^{-n} + a_{n-1} \cdot a^{-(n-1)} + \dots + a_0 = 0 \quad \Big| \cdot a^{n-1}$$

with $a_i \in A$

$$\rightsquigarrow a^{-1} + a_{n-1} + a_{n-2} \cdot a + \dots + a_0 \cdot a^{n-1} = 0$$

$$\Leftrightarrow a^{-1} = -\underbrace{\left(a_{n-1} + \dots + a_0 \cdot a^{n-1}\right)}_{\in A}$$

$\square$

Theorem 2.14. (Noether normalization)

Let K be a field, and let B be a finitely generated K-algebra. Then $\exists\, r \in \mathbb{N}$, elements $b_1, \dots b_r \in B$ which are algebraically independent s.t. B is finitely generated as $K[b_1, \dots, b_r]$-module. In other words, $\exists$ inj. K-alg. homomorphism $K[x_1, \dots, x_r] \hookrightarrow B$ s.t. B is f.g. module over $K[x_1, \dots x_n]$

Problem:

$\underbrace{K[y_1 .. y_n]}_{\text{alg. ind.}} \Rightarrow$

in general NOT independent

$$B = K[y_1 . y_n]/\mathfrak{a} = K[\overbrace{a_1 \dots, a_n}]$$

$$\exists \; \underbrace{K[b_1, \dots b_r]}_{\cong K[x_1 \dots x_r]}$$

$$\begin{tikzcd} B & \\ & K[b_1, \dots b_r] \arrow[ul] \\ K \arrow[uu, hook] \arrow[ur, hook] & \end{tikzcd}$$

f.g. module

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ex: $K[x] = K \oplus Kx \oplus Kx^2 \oplus \dots\dots$

$$K[x]/(f) = K \oplus \dots \oplus Kx^{n-1} \quad \deg(f) = n$$

Proof of 2.10. by 2.14.:

recall $L \supset K$ field extension where

L is f.g. K-alg.

to show: $L \supset K$ finite extension

apply 2.14: $\exists x_1, \dots, x_r \in L$ s.t.

int. domains

L is f.g. $K[x_1, \dots, x_r]$-module

L field $\xRightarrow{2.13} K[x_1, \dots, x_r]$ is field

$\Longrightarrow r = 0$

L is f.g K-module

$\Longleftrightarrow L$ f.g. K-vector space: $[L:K] < \infty$

$\square$