Recall: Theorem 2.14. (Noether normalization)

K field, B f.g. K-algebra. Then $\exists\, r \in \mathbb{N}_0$, $b_1, \dots, b_r \in B$ algebraically independent s.t. B is a finitely generated module over $K[b_1, \dots, b_r]$.

Proof: B f.g. K-algebra $\Rightarrow \exists$ some elements $c_1, \dots, c_n$ s.t. $B = K[c_1, \dots, c_n]$ (BUT: $c_1, \dots, c_n$ not alg. independent in general!)

If $c_1, \dots, c_n$ alg. independent $\leadsto r = n$, proof done.

If not: $\exists$ algebraic relation:

$$\boxed{\sum_{(\nu_1 \dots \nu_n) \in I} a_{\nu_1 \dots \nu_n} c_1^{\nu_1} \cdot \ldots \cdot c_n^{\nu_n} = 0} \quad (*)$$

where $a_{\nu_1 \dots \nu_n} \in K^{*}$, $I \subset \mathbb{N}_0^n$ finite idea: construct subalgebra of B which is generated (as K-algebra) by $< n$ Elements & such that B is a f.g. module over it.

Construction of subalgebra: Put

$$d_1 := c_1 - c_n^{s_1},\ d_2 := c_2 - c_n^{s_2}, \ldots, d_{n-1} := c_{n-1} - c_n^{s_{n-1}}$$

where $s_1, \ldots, s_n$ are exponents that will specified later

Clearly: $c_i = d_i + c_n^{s_i}$ $\quad i = 1, \ldots, n-1$

$$\rightsquigarrow B = K[c_1 \ldots c_n] = K[d_1, \ldots, d_{n-1}, c_n]$$

Aim: show that c_n is integral over $K[d_1, \dots, d_{n-1}]$ (for some choice of $(s_1, \dots, s_{n-1})$)

replace $c_1 \dots c_{n-1}$ in $(*)$ by $d_1 \dots d_{n-1}, c_n$:

$$\sum_{\underline{\nu} \in I} a_{\underline{\nu}} (d_1 + c_n^{s_1})^{\nu_1} \cdot \dots \cdot (d_{n-1} + c_{n-1}^{s_{n-1}})^{\nu_{n-1}} \cdot c_n^{\nu_n} = 0$$

multiply out:

$$\sum_{\underline{\nu} \in I} a_{\nu_1 \dots \nu_n} c_n^{s_1\nu_1 + \dots + s_{n-1}\nu_{n-1} + \nu_n} + \underbrace{g(d_1 \dots d_{n-1}, c_n)}_{\text{degree is } <} = 0$$

$$\max_{\underline{\nu} \in I} (s_1\nu_1 + \dots + s_{n-1}\nu_{n-1})$$

Claim: $s_1, \dots, s_{n-1}$ can be chosen s.t. $(\nu_1, \dots \nu_n) \neq (\nu_1', \dots, \nu_n')$

Then $s_1\nu_1 + \dots + s_{n-1}\nu_{n-1} + \nu_n \neq s_1\nu_1' + \dots + s_{n-1}\nu_{n-1}' + \nu_n'$

Pf. of claim: $N := \bigcup_{\nu \in I} \{\nu_1, \dots \nu_n\}$

$t := \max_{j \in N} (j) + 1$

Claim holds for

$s_1 := t^{n-1}, s_2 := t^{n-2}, \dots, s_{n-1} := t$

By claim: Different multi-indices $(\nu_1 \dots \nu_n) \in I$ give different degrees in c_n of $a_{\nu_1 \dots \nu_n} c_n^{s_1\nu_1 + \dots + s_{n-1}\nu_{n-1} + \nu_n}$,

hence, they don't cancel, write

M for the highest such degree and

a_M for its coefficient

$\rightsquigarrow$ relation: $\underset{\substack{\cap \\ K^\times}}{a} \cdot c_n^M + \underbrace{\tilde{g}(d_1, \dots d_{n-1}, c_n)}_{\text{degree in } c_n < M} = 0$

$\xrightarrow[\text{by } a]{\text{divide}}$ $c_n^M + \frac{1}{a} \tilde{g}(\dots) = 0$

integral relation for c_n over

$K[d_1, \dots, d_{n-1}]$

Since $B = \underbrace{K[d_1, \dots, d_{n-1}]}_{B'}[c_n]$

$= B'[c_n]$ with c_n integral over B'

$\overset{2.11}{\leadsto}$ B is f.g B'-module

B' is K-alg. generated by $n-1$ elements.

proceed by induction $\square$

This also finally proves all versions of Nullstellensatz

Some consequences of NS satz:

Lemma 2.15: K field, $(\lambda_1, \dots, \lambda_n) \in \mathbb{A}^n_K$

Then $m_\lambda := (x_1 - \lambda_1, \dots, x_n - \lambda_n) \subset R_n = K[x_1, \dots, x_n]$

is maximal.

Proof: Consider K-alg. homomorphism

$$\varphi_n : R_n \twoheadrightarrow K \quad \text{surj.}$$

$$x_i \longmapsto \lambda_i$$

$$K \ni \lambda \longmapsto \lambda$$

Then: $m_\lambda \subset \ker(\varphi_n) \rightsquigarrow \exists$ com. diagram

$$\begin{tikzcd} R_n \arrow[r, two heads, "\varphi_n"] \arrow[d, two heads] & K \\ R_n/m_\lambda \arrow[ur, "\tilde{\varphi}_n"'] & \end{tikzcd}$$

$\ker(\varphi) = m_\lambda$ $\Leftrightarrow$

<u>Claim</u>: $\tilde{\varphi}_n$ is injective: let $f \in R_n$ be given $\rightsquigarrow$ $\varphi_n(f)$ is obtained by replacing x_i by λ_i in f, i.e. it is "the value" $f(\lambda_1, \dots, \lambda_n)$.

If $f(\lambda_1, \dots, \lambda_n) = 0$ we need to show $f \in m_\lambda$

Pf by induction over n:

$n=1$: clear by polynomial division $\Big(f \in K[x], f(\lambda)=0$

$\Rightarrow f = (x-\lambda)\cdot g(x)$

$\rightsquigarrow f \in (x-\lambda) \subset R_1 \Big)$

suppose: $[f(\underline{\lambda})=0 \Rightarrow f \in m_\lambda]$ holds for all $f \in R_k$, $k<n$

Take $f \in R_n$, $f(\lambda_1, \dots \lambda_n)=0$, write

$f(x_1, \dots x_n) = (x_n - \lambda_n)\cdot g(x_1, \dots, x_n) + h(x_1, \dots, x_{n-1})$

$g \in R_n$, $h \in R_{n-1} \subset R_n$

$\Rightarrow h = f - (x_n - \lambda_n) \cdot g \in R_{n-1}$

$\Rightarrow h \in \ker(\varphi_n)$

$\Rightarrow h \in \ker(\varphi_n) \cap R_{n-1} = \ker(\varphi_{n-1})$

$\overset{\text{ind.}}{\underset{\text{hypoth.}}{\Longrightarrow}} \quad h \in (x_1 - \lambda_1, \dots, x_{n-1} - \lambda_{n-1})$

since $f = h + (x_n - \lambda_n) \cdot g \in m_\lambda$

$\rightsquigarrow \tilde{\varphi}_n$ bijective, $R_n/m_\lambda \cong K$ is field

$\Rightarrow m_\lambda$ maximal $\square$

Proposition 2.16.: Let K be a field, $R_n = K[x_1, \dots, x_n]$. Consider map

$$\Psi : \mathbb{A}^n_K \longrightarrow \{ m \subset R_n \mid m \text{ maximal} \}$$

$$(\lambda_1, \dots, \lambda_n) \longmapsto m_\lambda = (x_1 - \lambda_1, \dots, x_n - \lambda_n)$$

Then: 1.) Ψ injective

2.) $K = \overline{K} \Rightarrow \Psi$ surjective

Proof: 1.) Suppose $\lambda, \lambda' \in \mathbb{A}^n_K$ with $\boxed{m_\lambda = m_{\lambda'}}$ and $\boxed{\lambda \neq \lambda'}$, i.e. $\exists\, i \in \{1, \dots, n\}$: $\lambda_i \neq \lambda_i'$.

Then $x_i - \lambda_i \in m_1$, $x_i - \lambda_i' \in m_1$

$\underline{\lambda_i \neq \lambda_i'} \Longrightarrow$

$$\underbrace{\underbrace{\frac{1}{\underbrace{\lambda_i' - \lambda_i}_{\in K^\times}} \cdot \Big(\underbrace{(x_i - \lambda_i)}_{\in m_1} - \underbrace{(x_i - \lambda_i')}_{\in m_1}\Big)}_{\in m_1}}_{\frac{1}{\lambda_i' - \lambda_i}\left(\lambda_i' - \lambda_i\right) = 1}$$

Contradiction: $1 \in m_1$

since $m_1 \subsetneq R_n$

$\rightsquigarrow \varphi_1$ injective

2.) To show: $\forall\, m \subset R_n$ maximal, $\exists\, \lambda \in \mathbb{A}^n_K$ s.t. $m = m_\lambda$. Consider K-alg. homomorphism

$$K \hookrightarrow R_n \twoheadrightarrow R_n/m =: L$$

$L \supset K$ field extension (since $m \subset R_n$ maximal). $L = R_n/m$ is a f.g. K-alg. $\overset{2.10}{\Longrightarrow}$ $L \supset K$ finite field extension $\Rightarrow L/K$ algebraic

$\xRightarrow{K = \overline{K}}$ $L \cong K$

Write $\lambda_i := \bar{x} \in L \cong K$ for

the class of $x_i \in R_n$ in the quotient

$R_n / m = L \cong K$

$\Longrightarrow x_i - \lambda_i \in \ker (R_n \twoheadrightarrow L)$

$$\Longrightarrow \underbrace{(x_1 - \lambda_1, \ldots, x_n - \lambda_n)}_{\substack{\| \\ m_\lambda}} \subset m$$

Since both m_λ and m are maximal

in R_n, $m_\lambda \subset m$ implies $m_\lambda = m$

$\square$

§3. The Zariski topology, dimension of affine sets, the coordinate ring

Reminder on topological spaces

Def. 3.2. X set, $X \neq \emptyset$. Consider

$\mathcal{P}(X)$ power set $\mathcal{P}(X) = \{U \subset X\}$

Then $\tau \subset \mathcal{P}(X)$ is called topology on X iff

(T1) $\emptyset, X \in \tau$

(T2) If $(A_i)_{i \in I}$, $A_i \in \tau$ is family

($|I|$ can be ∞)

$$\Rightarrow \bigcup_{i \in I} A_i \in \tau$$

(T3) $A, B \in \tau \Rightarrow A \cap B \in \tau$

We call (X, τ) topological space

elements of τ are called open

sets. A subset $Y \subset X$ is called

closed $\Leftrightarrow X \setminus Y \in \tau$

<u>Def. 3.3.:</u> Let (X, τ) be top. space

Then $\tau' \subset \tau$ is called a basis of τ

iff $\forall U \in \tau : \exists (V_i)_{i \in I},\ V_i \in \tau'$

s.t. $U = \bigcup_{i \in I} V_i$

<u>Def 3.9</u>: A top. space (X, τ) is called Hausdorff if $\forall x, y \in X, x \neq y$ $\Rightarrow \exists A, B \in \tau$ s.t. $x \in A, y \in B$ s.t. $A \cap B = \emptyset$

<u>examples</u>: 1.) For any set X $\Rightarrow \exists$ discrete topology $\tau = \mathcal{P}(X)$ (i.e. any subset of X is open and also closed). This is a Hausdorff space (since for $x \neq y$, take $x \in \{x\}$, $y \in \{y\}$ $\Rightarrow \{x\}, \{y\} \in \tau$)

2.) Opposite example: $\tau = \{\emptyset, X\}$

i.e. no other open sets than $\emptyset, X$

clear: not Hausdorff.

3.) (X, d) metric space

is top. space by:

$$\tau_d := \{ U \subset X \mid \forall x \in U : \exists \varepsilon > 0 : B_\varepsilon(x) := \{ y \in X \mid d(x,y) < \varepsilon \} \subset U \}$$

ex: (X, τ_d) is Hausdorff.

Rk: Not every top. space is metric

Def. 3.5: (X, τ), (Y, σ) top. spaces

$f: X \to Y$. f is called continuous

$\Leftrightarrow \forall B \in \sigma : f^{-1}(B) \in \tau$.

If f bijective, and f^{-1} is contin.

then f is called homeomorphism.

X and Y are called homeomorphic

$\Leftrightarrow \exists f: X \xrightarrow{\cong} Y$ homeomorphism.

Def. 3.7.: Let X (X, τ) be top. space

then it is called quasi-compact

iff $\forall$ ~~coverings~~ $X = \bigcup_{i \in I} U_i$ by

open sets $U_i \subset X$ $U_i \in \tau$

$\exists$ finite subset $J \subset I$:

$$X = \bigcup_{i \in J} U_i$$

ex: $[0,1]$ quasi-compact

$(0,1]$ not — " — :

$\boxed{(0,1] = \bigcup_{n\in\mathbb{N}} (1/n, 1]}$ cannot be covered by finitely many subsets