Introduction to $\mathcal{D}$-modules

Overview of lecture: - The Weyl algebra, modules over it

- $\mathcal{D}$-modules on (smooth) affine varieties
- functors (direct / inverse image)
- holonomicity
- RH - correspondence

Introduction: Motivation from linear algebra

Consider system $\sum_{j=1}^{q} a_{ij} \, x_j = 0 \qquad i=1,\dots,p$

$a_{ij} \in K$. Matrix notation: $A = (a_{ij}) \in M(p \times q, K)$

$$\rightsquigarrow \quad K^p \xrightarrow{\varphi} K^q \qquad \varphi = - \cdot A$$

explicitly: $(y_1,\dots,y_p) \mapsto \left(\sum_{i=1}^{p} y_i \cdot a_{i1}, \dots, \sum_{i=1}^{p} y_i \cdot a_{iq}\right)$

Put $V := \operatorname{coker}(\varphi) \rightsquigarrow$ exact sequence

$$K^p \xrightarrow{\varphi} K^q \to V \to 0$$

Let $Sol((A,K) := \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix} \in K^q \,\middle|\, \sum_{j=1}^{q} a_{ij} x_j = 0 \,, \forall i \in \{1,..,p\} \right\}$

<u>„Theorem“:</u> $Sol(A,K) \simeq Hom_K(V,K)$

<u>„Proof“:</u> have $Hom_K(V,K) \simeq \{ f : K^q \to K \mid f \circ \varphi = 0 \}$

$$\begin{tikzcd}
K^p \arrow[r, "\varphi"] \arrow[dr, "0"'] & K^q \arrow[r] \arrow[d, "f"] & V \arrow[dl, dashed] \\
 & K &
\end{tikzcd}$$

Put $x_j := f(\overset{\overset{K^q}{\cup}}{e_j}) \qquad j = 1, \dots q$

$$\xRightarrow{f \circ \varphi = 0} f\Big(\overbrace{\sum_{i,j} y_i a_{ij} \cdot e_j}^{\text{general element in } im(\varphi)} \Big)$$

$$= \sum_{i,j} y_i a_{ij} \cdot x_j = \underset{q}{0} \quad \forall (y_1, .., y_p) \in K^p \iff$$

$$\Big(\sum_{j=1} a_{ij} x_j \Big)_{i=1..p} = \underline{0} \in K^p \iff \begin{pmatrix} x_1 \\ \vdots \\ x_q \end{pmatrix} \in Sol((A,K)$$

□

Remark: This also works for systems with $A \in M(p \times q, R)$, where $R = K[x_1, \dots x_n]$.

Notice: If S is R-algebra, e.g. $S = K[[x_1, \dots, x_n]]$)

$$\Rightarrow Sol(A, S) = \{ \underline{x} \in S^q \mid A \cdot \underline{x} = 0 \} \simeq Hom_R(coker(\varphi), S).$$

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generalize further to the non-commutative ring

$$A_n := A_n(K) := \left\{ P = \sum_{\substack{i_1 \dots i_n \\ \text{finite}}} f_{i_1 \dots i_n}(x_1, \dots x_n) \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}} \right\}$$

of linear partial differential operators with polynomial coefficients (i.e. $f_{i_1 \dots i_n} \in R$)

short-hand-notation: $P = \sum_{\substack{\alpha, \beta \in \mathbb{N}^n \\ \text{finite}}} c_{\alpha,\beta} \cdot x^\alpha \cdot \partial^\beta$

where: $c_{\alpha,\beta} \in K$; $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$; $\partial^\beta := \partial_1^{\beta_1} \dots \partial_n^{\beta_n} := \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$

A_n is called Weyl algebra. Multiplication.

$$[x_i, x_j] := x_i \cdot x_j - x_j \cdot x_i = 0$$

$$[\partial_i, \partial_j] = 0$$

$$[\partial_i, x_j] = \delta_{ij}$$ (remember $\partial_i(x_j f) = \partial_i(x_j) \cdot f + x_j \cdot \partial_i(f) = \delta_{ij} \cdot f + x_j \cdot \partial_i(f)$

$$\Rightarrow \partial_i(x_j f) - x_j \partial_i(f) = \delta_{ij} \cdot f$$

$$\Rightarrow \partial_i \cdot x_j - x_j \cdot \partial_i = \delta_{ij}$$

as operators acting on $K[\underline{x}]$)

notice: R is left A_n-module via

$$A_n \times R \longrightarrow R$$

$$(P, f) \longmapsto P(f)$$ (Differentiation of function f by P)

$\rightsquigarrow A_n \subset \mathrm{End}_K(R)$ subalgebra namely, smallest subalg. generated by R itself $(R \times R \to R, (r,f) \mapsto r \cdot f)$ and $\partial_1, \dots, \partial_n$.

Rk: This description also exists for R f.g. K-alg, not necessarily regular.

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description of linear systems of PDE's: let

$$\sum_{j=1}^{q} P_{i,j} u_j = 0 \qquad \forall i = 1, \dots, p$$

be given, where $P_{i,j} \in A_n$ and where we look for solutions $u_j \in S$, S a commutative K-algebra which is also A_n-module (e.g. $S = R$, $S = K[[x]]$ or, if $K = \mathbb{C}$, then $S = \mathbb{C}\{x\}$, or, if $K = \mathbb{R}$, $S = \mathcal{C}^\infty(\mathbb{R}^n)$, ...)

Description of $Sol((P, S) = \left\{ \begin{pmatrix} u_1 \\ \vdots \\ u_q \end{pmatrix} \in S^q \;\middle|\; \sum_{j=1}^{q} P_{i,j} u_j = 0 \right\}$

Put $\varphi : A_n^p \longrightarrow A_n^q$

$$(Q_1, \dots, Q_p) \longmapsto \left(\sum_{i=1}^{p} Q_i \cdot P_{ij} \right)_{j=1,\dots,q}$$

This is left A_n-linear (!). Put

$M := \operatorname{coker}(\varphi)$ → left A_n-module

again: diagram of left A_n-mod.

$$\begin{tikzcd} A_n^p \arrow[r, "\varphi"] \arrow[dr, "0"'] & A_n^q \arrow[r, "\pi"] \arrow[d, "f"] & M = A_n^q / \varphi(A_n^p) \arrow[r] \arrow[dl, dashed] & 0 \\ & S & & \end{tikzcd}$$

so $\operatorname{Hom}_{A_n}(M, S) = \{ f : A_n^q \to S \mid f \circ \varphi = 0 \}$

Put $\forall j = 1, .. q$: $u_j := f(e_j) \in S$ (determines f uniquely)

$$\Rightarrow \sum_{j=1}^{q} P_{ij} u_j = \sum P_{ij} f(e_j) = f\Big(\overbrace{\sum P_{ij} e_j}^{\in \operatorname{Im}(\varphi)} \Big) = 0 \quad \forall i = 1, \dots, p$$

Hence: $\operatorname{Sol}(P, S) \simeq \operatorname{Hom}_{A_n}(M, S)$

Example 1: $n=1$

- $M = A_1/A_1(x\partial_x - k)$ $k \in \mathbb{N}$ has sol'n $u = x^k \in R$
- $M = A_1/A_1(x\partial_x - k)$, $k \in \mathbb{Z}_{<0}$ has sol'n $u = x^k \in R[x^{-1}]$
- $M = A_1/A_1(x\partial_x - \alpha)$, $\alpha \notin \mathbb{Z}$ has sol'n $u = x^\alpha \notin R$ $\in \mathcal{O}(U^{\subset \mathbb{C}})$
- $M = A_1/A_1(\partial_x - 1)$ has sol'n $u = e^x \in \mathcal{O}(\mathbb{C})$
- $M = A_1/A_1(x^2\partial_x + 1)$ has sol'n $u = e^{\frac{1}{x}} \in \mathcal{O}(\mathbb{C}^*)$

first 3 ex. : regular singular

last 2 ex : irregular singular

example 2: still $n=1$, consider

2nd-order equation $a(x)u'' + b(x)u' + c(x)u = 0$

with $a, b, c \in R = K[x]$ $\Longleftrightarrow$ $M_1 := A_1/A_1(a \cdot \partial_x^2 + b \cdot \partial_x + c)$

Put $u_1 := u$, $u_2 := u'$ $\rightsquigarrow$ 1st-order system

$$u_1' - u_2 = 0$$

$$a \cdot u_2' + b \cdot u_2 + c \cdot u_1 = 0$$

$$\Longleftrightarrow \begin{pmatrix} \partial & -1 \\ c & a \cdot \partial_x + b \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow M_2 := \operatorname{coker}\left(A_1^2 \xrightarrow{\cdot \begin{pmatrix} \partial & -1 \\ c & a\partial_x + b \end{pmatrix}} A_1^2 \right)$$

<u>Claim</u>: (ex.) $\exists$ isomorphism of left A_1-modules

$$M_1 \xrightarrow{\cong} M_2$$

recall: Weyl algebra $A_n = A_n(K)$:

free K-algebra gen. by $x_1, \dots, x_n, \partial_1, \dots, \partial_n$

subject to relations $[x_i, x_j] = [\partial_i, \partial_j] = 0 \quad \forall i, j$

and $[\partial_i, x_j] = \delta_{ij}$

Philosophy: Homogeneous lin. systems of PDE's (with coeff.'s in R)

$\Longleftrightarrow$ fin. gen. left modules over A_n

Basic examples: (left) ideals in A_n (and quotients)

Prop.: A_n is simple as K-alg., i.e. $I \subset A_n$ left + right ideal $\Rightarrow I = (0)$ or $I = A_n$

Proof: Show (ex): The following rel. hold in A_n:

$$[\partial_i, x^\alpha \partial^\beta] = \alpha_i \cdot x^{\alpha - e_i} \partial^\beta \quad \text{and} \quad [x_j, x^\alpha \partial^\beta] = -\beta_j \cdot x^\alpha \partial^{\beta - e_j} \quad (*)$$

Suppose $(0) \subsetneq I \subset A_n$ is 2-sided ideal

Take $P = \sum_{\alpha,\beta} c_{\alpha,\beta} x^{\alpha} \partial^{\beta} \in I \setminus (0)$. Let

$$m_0 := \max \{ \alpha_1 \mid c_{\alpha,\beta} \neq 0 \}$$

supp. $m_0 > 0$, then $P_1 := [\partial_1, P] = \partial_1 P - P \partial_1 \in I$ (!!)

Def. m_1 as m_0 for $P_1 \xRightarrow{(*)} m_1 < m_0$

repeat until P_k with $m_k = 0$

Combine with $\partial_2, \dots, \partial_n \rightsquigarrow \exists\, Q \in I,\ Q \in K[\partial]$

Similarly $[Q, x_i] \in I \rightsquigarrow$ eliminate all x_i

…… $\Rightarrow \exists\, c \in I,\ c \in K^{*} \Rightarrow I = A_n$ □

Converting left into right modules & vice versa:

Lemma: (a) M left A_n-module $\Leftrightarrow$ M is R-module +

$d_i \in \mathrm{End}_K(M)$ with $[d_i, d_j] = 0$

& $d_i(x_j \cdot m) - x_j d_i(m) = \delta_{ij}\, m$ (*)

(b) M right A_n-mod. $\iff$ M is R-mod, $d_i \in \mathrm{End}_k(M)$

$[d_i, d_j] = 0$

& $d_i(x_j \cdot m) - x_j d_i(m) = -\delta_{ij} m$ (**)

Proof: (a) Put $\partial_i m := d_i(m)$ then $[\partial_i, \partial_j] m = 0$ and

$[\partial_i, x_j] m - \delta_{ij} m = 0$ $\rightsquigarrow$ M is left A_n-mod

(b) $m \partial_i := d_i(m)$ . . . . . . right A_n-mod

Consequence: Let M be a left A_n-mod.

Define commuting endom's $d_i \in \mathrm{End}_k(M)$ by

$d_i(m) := -\partial_i m$

$\Rightarrow$ M is R-mod & d_i's satisfy (**)

this yields right A_n-structure on M

concretely: If $P = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta \in A_n$, define

$P^T := \sum_{\alpha,\beta} c_{\alpha,\beta} \cdot (-1)^{|\beta|} \cdot \partial^\beta \cdot x^\alpha$ then $m . P := P^T \cdot m$

defines right A_n-structure

Filtrations on A_n and on A_n-modules:

For $P = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta \in A_n$, define

$$\operatorname{ord}(P) = \max\{|\beta| \mid c_{\alpha,\beta} \neq 0\}$$

Facts (ex:) - $\operatorname{ord}(P \cdot Q) = \operatorname{ord} P + \operatorname{ord} Q$

- $\operatorname{ord}[P,Q] = \operatorname{ord} P + \operatorname{ord} Q - 1$

formalize this:

Def: K field A an associative K-alg.

A filtration $F_\bullet A$ is a sequence of K-v.sp.

$$\{0\} = F_{-1}A \subseteq F_0 A \subseteq F_1 A \subseteq \ldots$$

s.t. $F_k A \cdot F_\ell A \subseteq F_{k+\ell} A$ and $\bigcup_{k \geq 0} F_k A = A$

(in part: $F_0 A \subseteq A$ is subalgebra & each $F_k A$ is a left + right $F_0 A$-module)

Example 1: $A = A_n$, $F_k^{\operatorname{ord}} A_n := \{P \in A_n \mid \operatorname{ord}(P) \leq k\}$

notice: $F_0^{ord} A_n = R = K[x]$ & $F_k A_n$ is f.g. R-mod

Example 2: still $A = A_n$,

$$F_k^B A_n := \left\{ P = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta \;\middle|\; |\alpha| + |\beta| \leq k \right\}$$

Bernstein filtration

here: $F_0^B A_n = K$, $F_k^B A_n$ is finite K-v.sp.

also: $F_k^B A_n \cdot F_e^B A_n = F_{k+e}^B A_n$

Notice: $F_\cdot^{ord}$ better suited when studying diff. systems on alg/analytic varieties

$F_\cdot^B$ only works for A_n but $\dim_K F_k^B < \infty$

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how to get commutative ring from $(A, F_\cdot A)$?:

Def: Let $(A, F_\cdot A)$ be a filtered ring, then define $gr_k^F A := F_k A / F_{k-1} A$ & $gr_\bullet^F := \bigoplus_{k \geq 0} gr_k^F A$

Then $\mathrm{gr}^F_\bullet A$ is graded K-algebra, with multiplication:

$$(r + F_{k-1}A) \cdot (s + F_{e-1}A) := (r \cdot s + F_{k+e-1}A)$$

$\forall r \in F_k A$, $s \in F_e A$ well-defined!

For order- resp. Berstein-filtration, we get:

<u>Lemma:</u> Put $A = A_n = A_n(K)$.

(a) $\mathrm{gr}^{ford}_\bullet A_n \simeq K[x_1, \dots, x_n, \bar{\partial}_1, \dots, \bar{\partial}_n]$

where $\bar{\partial}_i = \mathrm{Im}$ of ∂_i in F_1/F_0,

and $\deg(x_i) = 0$, $\deg(\bar{\partial}_i) = 1$

(b) $\mathrm{gr}^{FB}_\bullet A_n \simeq K[\bar{x}_1, \dots, \bar{x}_n, \bar{\partial}_1, \dots, \bar{\partial}_n]$

with $\deg(x_i) = \deg(\partial_i) = 1$

<u>Pf:</u> exercise (main point $[\partial_i, x_i] \in F_0$ in both cases)

<u>good filtrations:</u> <u>Definition</u>: Let M be a left A_n-module. A filtration $\{0\} = F_{-1}M \subseteq F_0 M \subseteq F_1 M \subseteq \dots$

by K-sub-vector spaces s.t. $M = \bigcup_{k \geq 0} F_k M$.
is called compatible with $(A, F_\cdot)$ if

- $\forall k, l \in \mathbb{N}$: $F_k A_n \cdot F_l M \subseteq F_{k+l} M$
- $\forall k \in \mathbb{N}$: $F_k M$ is f.g. $F_0 A$-module.

Put $gr_k^F M := F_k M / F_{k-1} M$ & $gr_\bullet^F M := \bigoplus_{k \geq 0} gr_k^F M$

Then $gr_\cdot^F M$ is a graded $gr_\bullet^F A_n$-module (check this)

Definition: $F_\bullet M$ is called a good filtration if $gr_\bullet^F M$ is a finitely generated $gr_\cdot^F A_n$-module.

Prop: $F_\bullet M$ is good $\iff \exists N \in \mathbb{N}: \forall l \geq N, \forall k \in \mathbb{N}$:

$$F_k A_n \cdot F_l M = F_{k+l} M$$

Proof: „$\Leftarrow$": We have $M = A_n \cdot F_N M$, hence

$$\mathrm{gr}^F_{\cdot} M = \mathrm{gr}^F_{\cdot} A_n \cdot \sum_{i=0}^{N} \mathrm{gr}^F_i M \,.$$

$F_i M$ is f.g. $F_0 A_n$-module $\Longrightarrow$ $\mathrm{gr}^F_i M$ is f.g. $\mathrm{gr}^F_0 A_n$-mod,

$\Longrightarrow$ $\mathrm{gr}^F_{\cdot} M$ is f.g. $\mathrm{gr}^F_{\cdot} A_n$-mod.

„$\Longrightarrow$“: suffices $\exists\, N : F_k A_n \cdot \overset{F_{N+1} M}{F_N M} \supset F_{k+N} M \quad \forall\, k \in \mathbb{N}$

(the statement follows then directly)

By assumption, $\mathrm{gr}^F_{\cdot} M$ has homogeneous system of generators as $\mathrm{gr}^F_{\cdot} A_n$-module, let $N :=$ max (deg of gen's)

$$\Longrightarrow \forall\, k \in \mathbb{N} : \mathrm{gr}^F_{k+N} M = \sum_{i=0}^{N} \mathrm{gr}^F_{k+N-i} A_n \cdot \mathrm{gr}^F_i M$$

$$\Longrightarrow F_{k+N} M = F_{k+N-1} M + \sum_{i=0}^{N} \underbrace{F_{k+N-i} A_n}_{\overset{\shortparallel}{F_k A_n \cdot F_{N-i} A_n}} \cdot F_i M$$

$$= F_{k+N-1} M + F_k A_n \cdot \sum_{i=0}^{N} F_{N-i} A_n \cdot F_i M \subset F_{k+N-1} M + F_k A_n \cdot F_N M$$

Then we obtain $F_{k+N} M \subset F_k A_n \cdot F_N M$ by induction on k $\square$

Corollary: M left A_n-mod. Then:

M has good filtration $\iff$ M is f.g. over A_n

Pf: exercise

Corollary: Let M left A_n-mod with good filtration $F_\cdot M$, let $G_\cdot M$ be a filtration compatible with $(A_n, F_\cdot A_n)$. Then $\exists N' \in \mathbb{N}$:

$$F_k M \subseteq G_{k+N'} M \quad \forall k \geq 0$$

Proof: Take $N \in \mathbb{N}$ with $F_k A_n \cdot F_N M = F_{k+N} M \quad \forall k \geq 0$

we have $F_N M = \sum_{i=1}^{r} F_0 A_n v_i$ and $M = \bigcup_e G_e M$. Hence

$\exists N'$: $v_1, \dots, v_r \in G_{N'} M \Rightarrow F_N M \subset G_{N'} M$

Then $\forall k$: $F_k M \subseteq F_{k+N} M = F_k A_n \cdot F_N M \subset$

$$F_k A_n \cdot G_{N'} M \subset G_{k+N'} M \qquad \square$$

In particular, if $F_\cdot M$ & $G_\cdot M$ are good $\Rightarrow$
$\exists N', N''$: $G_{k-N''} M \subset F_k M \subset G_{k+N'} M$

Proposition: M f.g. left A_n-mod.

$M' \subset M$ A_n-submodul $\Rightarrow M'$ f.g.

in particular: A_n is left (and right) noetherian.

Proof: M f.g $\Rightarrow \exists$ good filtration $F_\bullet M$

Put $F_k M' := F_k M \cap M'$, then

(a) $F_e A_n \cdot F_k M' \subset F_{k+e} M'$

(b) $F_k M'$ is f.g. $F_0 A_n$-mod (since $F_0 A_n = R$ which is noetherian and $F_k M' \subset F_k M$ which is f.g. over R) $\Rightarrow F_\bullet M'$ is compatible filtration

Moreover: $gr_\bullet^F M' \subset gr_\bullet^F M$ is $gr_\bullet^F A_n$-submodule

($gr_\bullet^F A_n$ noetherian, $F_\bullet M$ good, hence $gr_\bullet^F M$ f.g) $\Rightarrow gr_\bullet^F M'$ f.g.

hence $F_\bullet M'$ is good $\underset{\text{above}}{\overset{\text{cor.}}{\Longrightarrow}}$ M' is f.g. over A_n $\square$

recall $A_n = K[x_1,\dots,x_n]\langle\partial_1,\dots,\partial_n\rangle$ Weyl algebra

$$F_k^{ord} A_n = \{ P \in A_n \mid ord(P) \le k \}$$

$$F_k^{B} A_n = \{ \sum c_{\alpha\beta} x^\alpha \partial^\beta \in A_n \mid |\alpha| + |\beta| \le k \}$$

in this lecture: $F_\bullet A_n := F^B_\bullet A_n$

Let M be f.g. left A_n-module and choose good filtration $F_\cdot M$ (compatible with $F_\cdot A_n = F^B_\cdot A_n$).

$$\Rightarrow \dim_K F_k M < \infty \quad \forall k$$

Consider function

$$j \longmapsto \dim_K F_j M = \sum_{i=0}^{j} \overbrace{\dim \underbrace{gr_i^F M}_{F_iM / F_{i-1}M}}^{\dim F_iM - \dim F_{i-1}M}$$

Examples. 1.) $M = A_n$, $F_\bullet M = F_\bullet^B A_n$

$$\Rightarrow \dim \operatorname{gr}_k^F M = \dim K[\underline{x}, \underline{D}]_k = \binom{2n+k-1}{2n-1}$$

$$\Rightarrow \dim F_k M = \binom{2n+k}{2n}$$

$$= \frac{(2n+k)(2n+k-1)\cdot\ldots\cdot(k+1)}{(2n)!} \underset{k \gg 0}{\in} \mathbb{Q}[k]_{\le 2n}$$

2.) $M = K[x_1, \ldots x_n] = A_n / A_n(\partial_1, \ldots, \partial_n)$

$$F_k M = \{ f \in M \mid \deg_x f \le k \}$$

$$\rightsquigarrow \dim F_k M = \binom{n+k}{n} \in \mathbb{Q}[k]_{\le n} \quad \forall k \gg 0$$

3.) $M = K[\partial_1, \ldots, \partial_n] = A_n / A_n(x_1, \ldots x_n)$

$$\rightsquigarrow \dim F_k M = \binom{n+k}{n} \in \mathbb{Q}[k]_{\le n} \quad \forall k \gg 0$$

4.) $M = K[x, x^{-1}] = A_n \cdot x^{-1}$

pnf: $F_k M = F_k A_n \cdot x^{-1} = \bigoplus_{j=-k-1}^{k-1} K \cdot x^j$

$$\dim_K F_k M = 2k+1$$