<u>Recall</u>: $A_n = K[x_1,\dots,x_n]\langle\partial_1,\dots,\partial_n\rangle$ Weyl alg.

filtrations $F_\cdot A_n$, e.g.: $F^{ord}_\cdot A_n$, $F^B_\cdot A_n$ $\left(\Rightarrow \mathrm{gr}^F_\cdot A_n = K[x,\xi] \text{ comm.}\right)$

M f.g left A_n-module ($\Leftrightarrow$ system of lin. hom. PDE's)

$F_\cdot M$ good filtration $\Leftrightarrow$ 1.) compatible with $F_\cdot A_n$

2.) $\mathrm{gr}^F_\cdot M$ is f.g $F_\cdot A_n$-module

M f.g. A_n-mod. $\Leftrightarrow$ $\exists$ good filt. $F_\cdot M$.

$F_\cdot M$, $G_\cdot M$ good $\Rightarrow \exists N', N''$: $F_{k-N'}M \subset G_k M \subset F_{k+N''}M$

---

special case $F_\cdot A_n := F^B_\cdot A_n$ $\left(\Rightarrow \dim_K F_k A_n < \infty \; \forall k\right)$

If $F_\cdot M$ is good filtration (hence, compatible with $F_\cdot A_n = F^B_\cdot A_n$), then also $\dim_K F_k M < \infty$. Then we have the following

<u>Proposition</u>: $\exists\, \chi^{(M,F.M)} \in \mathbb{Q}[t]_{\leq 2n}$

s.t. $\dim_k F_k M = \chi^{(M,F.M)}(k) \quad \forall k \gg 0$.

<u>Proof</u>: Use that $S := gr_\bullet^F A_n \simeq K[\underline{x}, \underline{D}]$ is commutative and that $gr_\bullet^F M$ is a graded S-module. Then we have:

Hilbert's syzygy theorem: $\exists$ resolution of $gr_\bullet^F M$ of length $2n$ by graded free S-modules:

$$0 \to E_{2n} \to E_{2n-1} \to \ldots \ldots \to E_1 \to E_0 \to gr_\bullet^F M \to 0$$

where $E_p = \bigoplus_{\substack{q \geq 0 \\ \text{finite}}} S(-q)^{b_{p,q}}$ with $S(-q)_i := S_{i-q}$

recall that $\dim_K S_i = \binom{2n+i-1}{2n-1}$

$$\Rightarrow \dim_K \mathrm{gr}^F_k M = \sum_{p=0}^{2n} (-1)^p \sum_q b_{p,q} \cdot \binom{2n+k-q-1}{2n-1}$$

$$\Rightarrow \dim F_j M = \sum_{k=0}^{j} \underbrace{\qquad\qquad}$$

$$= \sum_{p=0}^{2n} (-1)^p \sum_q b_{p,q} \underbrace{\sum_{k=0}^{j} \binom{2n+k-q-1}{2n-1}}_{\binom{2n+j-q}{2n} \in \mathbb{Q}[j]_{2n}}$$

$$\in \mathbb{Q}[j]_{\leq 2n} \quad \forall\, k \gg 0 \qquad \square$$

Let $M \neq 0$, put $d := \deg \chi^{(M,F,M)} \geq 0$

<u>ex</u>: $\chi^{M,F,M}(t) = \frac{m}{d!} \cdot t^d + \cdots\cdots$

<u>Lemma:</u> d and m depend only on M, not on choice of $F_\bullet M$.

<u>Proof:</u> Let $F_\bullet M$, $G_\bullet M$ be good filtrations compatible with $F_\bullet A_n = F^B_\bullet A_n$, then $\exists k \geq 0$

$$F_{j-k} M \subseteq G_j M \subseteq F_{j+k} M \quad \forall j \geq 0$$

$$\Rightarrow \dim_K F_{j-k} M \leq \dim_K G_j M \leq \dim_K F_{j+k} M$$

$$\Rightarrow \chi^{(M, F_\bullet M)}(j-k) \leq \chi^{(M, G_\bullet M)}(j) \leq \chi^{(M, F_\bullet M)}(j+k)$$

$\forall j \gg 0$. The 3 polynomials

$$\chi^{(M, F_\bullet M)}(t-k), \quad \chi^{(M, F_\bullet M)}(t), \quad \chi^{(M, F_\bullet M)}(t+k)$$

have the same leading term $\frac{m}{d!} t^d$

$\Longrightarrow$ hence $\chi^{(M, G_\bullet M)}(t)$ has l.t. $\frac{m}{d!} t^d$ $\square$

Def.: d is called the dimension of M

m — " — multiplicity of M

If $M \neq 0 \Rightarrow d(M) \geq 0$ & $m \geq 1$

examples: - $M = A_n \Rightarrow d = 2n$, $m = 1$

- $M = K[x]$ or $M = K[\partial] \Rightarrow d = n$, $m = 1$

- $M = K[x, x^{-1}] \Rightarrow d = 1$, $m = 2$

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need to study behavior of d, m for sub-/quotient mod's:

If $0 \to M' \to M \to M'' \to 0$ is short exact sequence of A_n-modules, and if $F_\bullet M$ is any filtration, then put $\forall k \in \mathbb{N}$:

$F_k M' := F_k M \cap M'$ and $F_k M'' := \mathrm{Im}(F_k M)$ in M''

ex: $\forall k$, we have short exact sequences of A_n- resp. of $gr_\cdot^F A_n$- modules

$$0 \to F_k M' \to F_k M \to F_k M'' \to 0$$

resp.: $0 \to gr_k^F M' \to gr_k^F M \to gr_k^F M'' \to 0$

Prop: Let M be f.g. over A_n, and let $F_\cdot M$ be good (always w.r.t. $F^B A_n$), then both $F_\cdot M'$, $F_\cdot M''$ are good and

1.) $\chi^{(M, F_\cdot M)}(t) = \chi^{(M', F_\cdot M')}(t) + \chi^{(M'', F_\cdot M'')}(t)$

2.) $d(M) = \max\{d(M'), d(M'')\}$

3.) If $d(M') = d(M'') \Rightarrow m(M) = m(M') + m(M'')$

Proof: $F_\cdot M$ good $\Rightarrow gr_\cdot^F M$ is f.g. over $gr_\cdot^F A_n = K[\underline{x}, \underline{\xi}]$

$\xrightarrow[\text{noetherian}]{K[x,\xi]}$ $gr_\cdot^F M'$, $gr_\cdot^F M'$ f.g. (i.e. $F_\cdot M'$, $F_\cdot M''$ good)

$$\Rightarrow 0 \to gr_\cdot^F M' \to gr_\cdot^F M \to gr_\cdot^F M'' \to 0$$

is short exact sequence of f.g. $gr_\cdot^F A_n$-mod.

$$\Rightarrow \dim_K gr_k^F M = \dim_K gr_k^F M' + \dim_K gr_k^F M'' \quad \forall k$$

$$\Rightarrow \dim_K F_k M = \dim_K F_k M' + \dim_K F_k M'' \quad \forall k$$

$$\Rightarrow \chi^{(M, F_\cdot M)}(t) = \chi^{(M', F_\cdot M')}(t) + \chi^{(M'', F_\cdot M'')}(t)$$

$\square$

Example: $M = K[x, x^{-1}]$, then $\exists$ short exact seq. of A_1-mod

$$0 \to K[x] \to K[x, x^{-1}] \to K[x]/K[x,x^{-1}] \to 0$$

notice: $K[x,x^{-1}]/K[x]$ is generated over A_1 by x^{-1}, but we also have $x \cdot x^{-1} = 0$ in M, hence:

$K[x,x^{-1}]/K[x] \simeq A_1/A_1 \cdot x$. We have seen

$d(K[x]) = d(A_1/A_1 x) = m(K[x]) = m(A_1/A_1 x) = 1$

hence $d(K[x,x^{-1}]) = 1$ & $m(K[x,x^{-1}]) = 2$

as can be directly calculated (done last time)

<u>Theorem (Bernstein's inequality)</u>

Let $M \neq 0$ be a f.g. A_n-module, then

$$d(M) \geq n$$

For the proof, we can fix good filtration $F_\bullet M$ (since $d(M)$ does not depend on it), and by shifting we assume $F_0 M \neq 0$.

<u>Lemma 3.6.:</u> $\forall j \in \mathbb{N}$, the map

$$F_j^B A_n \longrightarrow \mathrm{Hom}_K(F_j M, F_{2j} M)$$

$$P \longmapsto (m \longmapsto P \cdot m)$$

is injective

Proof of the lemma: induction on j:

$j=0$: $F_0 A_n = K$, $F_0 M \neq 0 \rightsquigarrow K \hookrightarrow \mathrm{Hom}_K(F_0 M, F_0 M)$

$j>0$: suppose $\exists\, P \in F_j A_n \cap \ker(F_j M \to F_{2j} M)$

$\Rightarrow P$ non-constant (since $F_j M \neq 0$)

$\rightsquigarrow P$ contains some x_i or some ∂_i

supp. it contains $x_i \Rightarrow [P, \partial_i] \neq 0$

clearly $[P,\partial_i] \in F_{j-1} A_n$ and $\forall\, m \overset{\neq 0}{\in} F_{j-1} M$:

$[P,\partial_i] m = P(\partial_i m) - \partial_i P(m) = 0$ since $m, \partial_i m \in F_j M$

↯ ind. hyp: $\forall\, Q \in F_{j-1} A_n$, $Pm \neq 0$ $\forall\, m \in F_{j-1} M \setminus \{0\}$.

If P contains ∂_i, use same argument with

$[P, x_i]$ instead $\square$

<u>Proof of theorem:</u> By the lemma:

$$\dim_K F_j^B A_n \leq \dim_K \mathrm{Hom}_K(F_j M, F_{2j} M) =$$

$$\dim_K F_j M \cdot \dim_K F_{2j} M$$

$$\Rightarrow \forall j \gg 0: \binom{j+2n}{2n} \leq \chi^{(M, F.M)}(j) \cdot \chi^{(M, F.M)}(2j)$$

$$\begin{array}{ccc} \binom{j+2n}{2n} & \chi^{(M,F.M)}(j) & \chi^{(M,F.M)}(2j) \\ \downarrow {\scriptstyle j \gg 0} & \downarrow {\scriptstyle j \gg 0} & \downarrow {\scriptstyle j \gg 0} \\ \in \mathbb{Q}[j]_{2n} & \in \mathbb{Q}[j]_d & \in \mathbb{Q}[j]_d \end{array}$$

$$\Rightarrow 2n \leq 2d \Rightarrow n \leq d \qquad \square$$

<u>Definition:</u> Let M be a f.g. A_n-module. Then if either $M=0$ or $d(M)=n$, M is called a holonomic A_n-module.

Corollary: If $0 \to M' \to M \to M'' \to 0$ is short exact sequence of A_n-modules. Then M is holonomic $\Longleftrightarrow$ M' and M'' are holonomic.

Proof: exercise

Corollary: M holonomic $\Rightarrow$ M is noetherian + artinian, i.e. every descending sequence

$$M = M_0 \supsetneq M_1 \supsetneq \dots$$ is stationary.

Moreover, $\exists$ finite sequence $M = M_0 \supset M_1 \supset \dots \supset M_e$ where each M_i / M_{i+1} is simple

Sketch of proof: Let $\{0\} = N_0 \subsetneq N_1 \subsetneq \dots \subsetneq N_k = M$ be any increasing sequence. M hol. $\Rightarrow$ N_i hol. (last corollary), hence $d(N_i) = n$. Then

$$m(M) = m(N_1) + m(M/N_1) = m(N_1) + m\left(N_2/N_1\right) + m\left(M/N_2\right)$$
$$= \dots = m(N_1) + m\left(N_2/N_1\right) + m\left(N_3/N_2\right) + \dots + m\left(N_k/N_{k-1}\right)$$

Since $m(N_i/N_{i-1}) \geq 1 \Rightarrow m(M) \geq k$.

Hence, any strictly decreasing sequence has length $\leq m(M) \Rightarrow M$ is artinian.

2nd statement: exercise

also: give example for infinite sequence for non-holonomic M

$\square$

important example for holonomic modules:

<u>Prop.</u>: Let $0 \neq p \in K[x_1, \dots, x_n]$, then $M := K[x_1, \dots, x_n][p^{-1}]$ has structure of holonomic left A_n-module.

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notice that a priori, we don't even know that M is finitely generated over A_n