Applications of holonomicity

Recall from last time: M f.g. left A_n-module

$\rightsquigarrow d(M) \in [n, 2n]$ dimension of M, $d(M) = n \iff$

M is holonomic; holonomic modules

are artinian, i.e. every $M = M_0 \supsetneq M_1 \supsetneq \dots$

becomes eventually stationary

<u>Prop.:</u> Let $0 \neq p \in K[x_1, \dots, x_n]$, then

$M := K[x_1, \dots, x_n][p^{-1}]$ has structure of

holonomic left A_n-module.

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notice that a priori, we don't even know

that M is finitely generated over A_n

For the proof, we have holonomicity criterion

Lemma: M A_n-mod, $F_\cdot A$ compatible (with $F^B_\cdot A_n$) filtration s.t. $\dim_k F_k M \le \frac{c}{n!} k^n + c'(k+1)^{n-1}$ for some $c, c' \ge 1$ $\Longrightarrow$ M holonomic, $m(M) \le c$.

In particular, M is A_n-f.g.

<u>Proof of Prop. using Lemma:</u>

A_n-structure on $M = K[x][p^{-1}]$:

$$\partial_k (f p^{-\ell}) = -\ell f \frac{\partial p}{\partial x_k} p^{-\ell-1} + \frac{\partial f}{\partial x_k} p^{-\ell}$$

$$= \left(-\ell f \frac{\partial p}{\partial x_k} + p \frac{\partial f}{\partial x_k}\right) p^{-(\ell+1)}$$

Let $m := \deg(p)$, put $F_k M := \{ f \cdot p^{-e} \mid \deg(f) \le (m+1)e \}$

<u>check:</u> – $\dim_K F_k M < \infty$

– $f p^{-\ell} \in F_k M \Rightarrow x_j \cdot f p^{-\ell},\ \partial_j (f p^{-\ell}) \in F_{k+1} M$

– $\bigcup_p F_p M = M$

moreover: $\dim_K F_k M = \binom{(m+1)k+n}{n} \in \mathbb{Q}[k]_{\leq n} \quad \forall k \gg 0$

$$= \frac{((m+1)k+n)((m+1)k+n-1)\cdot\ldots\cdot((m+1)k+1)}{n!}$$

leading coeff: $\frac{(m+1)^n}{n!} \xRightarrow{\text{Lemma}} M$ hol., $m(M) \leq (m+1)^n$

$\square$

Proof of the lemma:

Claim 1: Let $0 \neq N \subset M$ with N f.g. (!!)

$\Rightarrow N$ is holonomic:

Let $G_\bullet N$ be good filtration on N

consider $F_\bullet M \cap N$ (not necessarily good)

$\xRightarrow[\text{lecture}]{\text{last}} \exists k: \; G_j N \subseteq F_{j+k} M \cap N \subseteq F_{j+k} M$

$\Rightarrow \dim_K G_j N \leq \dim_K F_{j+k} M \leq \frac{c}{n!}(j+k)^n + c'(j+k+1)^{n-1}$

Both sides are pol's in j $\forall j \gg 0$

$$\Rightarrow \deg \chi^{(N, G_\bullet N)}(t) \le n$$

Bernstein $\Rightarrow$ $= n$

$\Rightarrow N$ holonomic & $m(N) \le c$

<u>Claim 2:</u> M is f.g., hence by applying Claim 1 $\Rightarrow M$ hol., $m(M) \le c$

Take $0 \neq m_1 \in M$, $N_1 := A_n m_1 \subset M$

$N = M \Rightarrow$ ✓

otherwise, take $m_2 \in M \setminus N_1$, $N_2 := A_n m_1 + A_n m_2$

$\Rightarrow \ldots$ chain: $N_1 \subset N_2 \subset N_3 \subset \ldots \subset M$

since $m(N_i) \le c$ $\forall i$

$\Rightarrow$ chain stabilizes after $\le c$ steps $\square$

Application to analysis

remark on distributions & right A_n-modules

Let $\mathcal{F}$ be some space of (test) functions (with an appropriate topology), and let $\mathcal{F}'$ be its topological dual. Suppose that $\mathcal{F}$ is a left A_n-module, then $\forall\, T \in \mathcal{F}'$, $P \in A_n$, we put

$$(T.P)(\varphi) := T(P.\varphi) \quad \forall \varphi \in \mathcal{F}$$

<u>exercise</u>: this defines structure of right A_n-module on $\mathcal{F}'$

<u>example</u>: $\mathcal{F} = \mathcal{C}^\infty(\mathbb{R}^n)$ (topology: uniform convergence of all derivatives on $K \subset \mathbb{R}^n$ compact)

then $\delta_0 \in \mathcal{F}'$: Delta distribution $\delta_0(\varphi) = \varphi(0)$

<u>exercise</u>: - check $\delta_0 \cdot x_i = 0$

- submodule $\delta_0 \cdot A_n \subset \mathcal{F}'$ is isomorphic to $A_n / (x_1, \dots, x_n) A_n \simeq \mathbb{R}[\partial_1, \dots, \partial_n]$

Consider also Schwarz space

$$S(\mathbb{R}^n) := \left\{ \varphi \in C^\infty(\mathbb{R}^n) \,\middle|\, \sup_{x \in \mathbb{R}^n} \left| \left(x^\alpha \partial_x^\beta \varphi\right)(x) \right| < \infty \right\}$$
$$\forall \alpha, \beta \in \mathbb{N}^n$$

$S(\mathbb{R}^n)$ again top. vector space (w.r.t. semi-norms $\|\varphi\|_k = \sup_{x \in \mathbb{R}^n} \max_{|\alpha|, |\beta| < k} \left| \left(x^\alpha \partial_x^\beta \varphi\right)(x) \right|$)

<u>example:</u> $e^{-x^2} \in S(\mathbb{R})$ since it rapidly decreases at $\pm\infty$ (as well as all derivatives)

<u>We call $T \in S'(\mathbb{R}^n)$ a tempered distribution</u>

Let now $p \in \mathbb{R}[x_1, \dots, x_n]$ with $p(x) \geq 0$ $\forall x \in \mathbb{R}^n$ be given and consider $\forall s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$

$$T_s(\varphi) := \int_{\mathbb{R}^n} p(x)^s \varphi(x) \, d\mu(x)$$

here $p(x)^s := \begin{cases} 0 & \text{if } p(x)=0 \\ e^{s\cdot \ln p(x)} & \text{if } p(x)>0 \end{cases}$

ex: $T_s(\varphi)<\infty$ & $s\mapsto T_s(\varphi)$ is holom.

Question (Gelfand' ICM 1954)

$\exists$ meromorphic continuation of $T_s(\varphi)$? related to problem of division of distributions: $\exists ?\, T\in S'(\mathbb{R}^n): T\cdot p = 1$?

Th (Bernstein '72): $\exists\, b\in\mathbb{R}[s]$, $P\in A_n(\mathbb{R})[s]$ s.t. $P\bullet p^{s+1} = b(s)\cdot p^s$ (Bernstein's functional equation)

$$\Rightarrow b(s)\cdot T_s(\varphi) = \int_{\mathbb{R}^n} b(s)\, p(x)^s\, \varphi(x)\, d\mu(x)$$

$$= \int_{\mathbb{R}^n} P(x,\partial_x,s)\, p(x)^{s+1}\varphi(x)\, d\mu(x) \underset{\text{int. by parts}}{=} \underbrace{\int_{\mathbb{R}^n} p^{s+1}\, P^{tr}(x,\partial_x,s)\,\varphi(x)\, d\mu}_{\in\, \mathcal{O}(\mathbb{R}_{>0}+i\mathbb{R})}$$

$$\Rightarrow T_s(\varphi) := \frac{1}{b(s)}\cdot \text{RHS} \in \mathcal{O}(\mathbb{R}_{>-1}+i\mathbb{R})\,(\setminus b^{-1}(0))$$

$$\Rightarrow \ldots\ldots\; T_s(\varphi)\in \mathcal{O}(\mathbb{C})\,(b^{-1}(0)+\mathbb{Z}_{\le 0})$$

Proof of existence of Bernstein's functional equation:

Let now K be arbitrary, $p \in K[x_1, \dots, x_n]$ polynomial $p \neq 0$. Recall that $K[\underline{x}, p^{-1}]$ is left A_n-holonomic

Generalization: Consider field $K(s) = Q(K[s])$ and $K(s)$-v.sp. $M = K(s)[x_1, \dots, x_n, p^{-1}]$ endowed with the following left action: We put

$$\partial_j (f \cdot p^{-\ell}) = \frac{\partial f}{\partial x_j} p^{-\ell} + (s-\ell) \cdot f \cdot \frac{\partial p}{\partial x_j} p^{-(\ell+1)}$$

Trick to remember this formula: Introduce formal symbol p^s, where $\partial_j p^s = s p^{-1} \frac{\partial p}{\partial s} \cdot p^s$ and consider M as the free $K[\underline{x}, p^{-1}]$-module of $rk = 1$ generated by p^s. Then elements of M are of the form $m = f \cdot p^{s-\ell}$, $f \in K(s)$, and we have formal diff. rul:

$$\partial_j m = (\partial_j f) \cdot p^{s-\ell} + (s-\ell) \cdot f \cdot (\partial_j p) \cdot p^{s-(\ell+1)}$$

exercise 1: Check that this defines left $A_n(K(s))$-action!

Put, as before

$$F_k M := \left\{ f p^{s-\ell} \;\middle|\; \underbrace{\deg(f)}_{} \leq \left(\overset{\deg p}{\overset{\|}{m}}+1\right) \cdot \ell \right\}$$

exercise 2: Check that this defines filtration compatible with Bernstein filtration on $A_n(K(s))$ (notice: $\deg(s) = 0$).

exercise 3: Compute that

$$\dim_{K(s)} F_k M = \binom{(m+1)\cdot k + n}{n} = \frac{((m+1)k+n)\cdot \ldots \cdot (m+1)k+1}{n!}$$

This is pol. in k of deg

$$(m+1)k + n + 1 - (m+1)k + 1 = n$$

with leading coefficient $(m+1)^n$.

Lemma above $\Rightarrow$ M is holonomic and

$m(M) \leq (m+1)^n$.

Now consider for any $r \geq 0$ the submodule:

$$M_r := A_n(K(s)) \cdot p^r = A_n(K(s)) \cdot p^{s+r} \subset M$$

we have

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots$$

M holonomic $\Rightarrow$ each M_r is holonomic

$\Rightarrow$ chain must stabilize after $\leq m(M)$ steps

$\Rightarrow \exists\, r: \quad M_r \subset M_{r+1} \Rightarrow p^{s+r} \in M_{r+1} = A_n(K(s))\, p^{s+r+1}$

$$\exists\, Q(s) \in A_n(K(s)) : Q(s)\, p^{s+r+1} = p^{s+r}$$

Let $d(s) \in K[s]$ s.t. $R(s) := d(s) \cdot Q(s) \in A_n(K)[s]$.

$$\Rightarrow R(s) \cdot p^{s+r+1} = d(s) \cdot p^{s+r}$$

replace s by $s-r$ (exercise 4: check that this is compatible with $A_n(K(s))$-structure on M): $R(s-r)\cdot p^{s+1} = d(s-r)\cdot p^s$

This yields existence of functional equation.

Exercise 5: $\{ b(s) \in K[s] \mid \exists P \in A_n(K)[s]: P \cdot p^{s+1} = b(s)\cdot p^s \}$

is ideal in $K[s]$

Def.: The monic polynomial of min. degree $b(s)$ satisfying $P\cdot p^{s+1} = b(s)\cdot p^s$ for some $P \in A_n(K)[s]$ is called the Bernstein-Sato-polynomial (or global b-function) of p and denoted by $b_p(s)$.

examples: - $p = x \Rightarrow \partial_x (x^{s+1}) = (s+1) \cdot x^s$

$\Rightarrow b_p(s) = s+1$

- $p = x^2 \Rightarrow \partial_x^2 \left(x^{2(s+1)}\right) = \partial_x \left(2(s+1)\, x^{2s+1} \right)$

$= 2(s+1)(2s+1)(x^2)^s$

$\Rightarrow b_p = (s+1)(s+\frac{1}{2})$

- $p = x^k \Rightarrow b_p(s) = (s+\frac{1}{k}) \cdot (s+\frac{2}{k}) \cdot \ldots \cdot (s+\frac{k-1}{k}) \cdot (s+1)$

- $p = x_1 \cdot \ldots \cdot x_k \Rightarrow (\partial_{x_1} \cdot \ldots \cdot \partial_{x_k})\, p^{s+1} = (s+1)^k p^s$

$\Rightarrow b_p(s) = (s+1)^k$

- $p = \prod_{i=1}^{k} x_i^{n_i} \Rightarrow \left(\prod_{i=1}^{k} \partial_{x_i}^{n_i} \right) p^{s+1} = \prod_{i=1}^{k} \prod_{j_i=1}^{n_i} \left(s + \frac{j_i}{n_i}\right) p^s$

- $p = x_1^2 + \ldots + x_k^2$

$\Rightarrow \partial_{x_i}^2 p^{s+1} = \partial_{x_i} \left[(s+1)\, p^s\, \partial_{x_i} p \right] = \partial_{x_i} \left[(s+1)\, p^s \cdot 2x_i \right]$

$= 2(s+1)\, \partial_{x_i} [x_i \cdot p^s] = 2(s+1)\left(p^s + x_i \cdot s \cdot p^{s-1} \cdot 2x_i\right)$

$= 2(s+1)\left(p^s + 2x_i^2 s p^{s-1}\right)$

$$\Rightarrow \left(\sum_{i=1}^{k} \partial_{x_i}^2\right) p^{s+1} = 2(s+1) \sum_{i=1}^{k} \left(p^s + 2x_i^2 s p^{s-1}\right)$$

$$= 2(s+1)\left(k \cdot p^s + 2 \cdot \underbrace{\left(\sum_{i=1}^{k} x_i^2\right)}_{=p} \cdot s \cdot p^{s-1}\right)$$

$$= 2(s+1)(k+2s)\, p^s$$

$$= 4(s+1)\left(s+\frac{k}{2}\right) p^s$$

$$\Rightarrow b_p(s) = (s+1)\cdot\left(s+\frac{k}{2}\right)$$

Facts: -1 is always root of $b_p(s)$

(Clear: $P \cdot p^0 = b_p(-1) \cdot p^{-1}$ only holds if $b_p(-1) = 0$)

- (Kashiwara - Malgrange): roots of $b_p(s) \subset (-n, 0) \cap \mathbb{Q}$

- Kashiwara: $b_p(\alpha) = 0 \iff$

$e^{-2\pi i \alpha}$ is EV of mon. operator

on $H^*(p^{-1}(t), \mathbb{C})$

- (...): p quasi-hom. (i.e. $\exists\, w_1, \dots w_n \in \mathbb{Q}_{>0}$

$p = \sum_{\underline{\alpha}} c_{\underline{\alpha}}\, x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ s.t. $\sum_{i=1}^{n} \alpha_i w_i \neq 1 \Rightarrow c_{\underline{\alpha}} = 0$)

with isolated crit. point at $\underline{x} = \underline{0}$

$\Rightarrow$ roots expressable by w_i's

e.g. $p = x_1^2 + x_2^3$ $\left(w_1 = \frac{1}{3},\ w_2 = \frac{1}{2}\right)$

$$\Rightarrow \left(\frac{1}{27}\partial_{x_2}^3 + \frac{x_2}{6}\partial_{x_1}^2\partial_{x_2} + \frac{x_1}{8}\partial_{x_1}^3\right) p^{s+1}$$

$$= \underbrace{\left(s + \frac{5}{6}\right)(s+1)\left(s + \frac{7}{6}\right)}_{= b_p(s)} p^s$$

(roots symm. around -1 !)