Characteristic varieties

recall: M f.g. left A_n-module $\rightsquigarrow \exists$ good filtration $F_\bullet M$ (compatible with $F_\bullet^B A_n$)

$\rightsquigarrow \chi^{(M, F_\bullet M)}(t)$ Hilbert pol of $gr_\bullet^F M \rightsquigarrow d(M), m(M)$

recall from commutative algebra: P graded module over graded ring $S \overset{=K[y_1 \dots y_m]}{}$ $\Rightarrow \exists\, \chi^P(t) \in \mathbb{Q}[t]$ with $\chi^P(k) = \dim_K P_k \quad \forall k \gg 0$ and

$\deg \chi^P(t) = \dim^{Krull}(\operatorname{Supp} P)$, recall

$$\operatorname{supp}(P) = \{ \underline{y} \in K^m \mid P_{m_y} \neq 0 \}$$
$$\| $$
$$V(\sqrt{\operatorname{ann} P})$$

where $\operatorname{ann} P = \{ f \in S \mid f \cdot m = 0 \ \forall m \in P \}$

back to M f.g. left A_n-module

Consider order filtration $F_\bullet^{ord} A_n$ and chose compatible filtration $F_\bullet M$, define

$$I(M, F_\bullet M) := \mathrm{ann}_{gr^{F_\bullet} A_n}(gr^{F_\bullet} M)$$

and $J(M) := \sqrt{I(M, F_\bullet M)}$

<u>Lemma</u>: $J(M)$ does not depend on choice of $F_\bullet M$ (compatible with $F_\bullet^{ord} A_n$)

<u>Proof</u>: Notation: $P \in F_k A_n$, $m \in F_j M$

$\Rightarrow [P] \in gr_k^F A_n$ (principal) symbol of P

$[m] \in gr_j^F M$

$\rightsquigarrow [P] \cdot [m] = [P \cdot m] \in gr_{j+k}^F M$

hence $[P]\cdot[m]=0 \iff P\cdot m \in F_{j+k-1}M$

hence $\underline{I}(M,F_\cdot M) = \mathrm{ann}_{\mathrm{gr}^F A_n}\mathrm{gr}^F M = \langle [P]\in \mathrm{gr}_k A_n \mid P\cdot F_j M \subset F_{k+j-1}M\ \forall j\geq 0\rangle$

$\Uparrow$ homogeneous ideal

$\rightsquigarrow J(M) = \sqrt{\underline{I}(M,F_\cdot M)} = \langle [P]\in \mathrm{gr}_k A_n \mid \exists m\geq 0$

$P^m \cdot F_j M \overset{(*)}{\subset} F_{mk+j-1}M\ \forall j\rangle$

Let $G_\cdot M$ be another good filtration on M (compatible with $F_\cdot^{\mathrm{ord}}A_n$), then we know: $\exists j_0\geq 0$

$$F_j M \subset G_{j+j_0}M \qquad G_j M \subset F_{j+j_0}M$$

Take $[P]\in\sqrt{\underline{I}(M,F_\cdot M)} \rightsquigarrow \exists m\geq 1: P^m F_j M \subset F_{mk+j-1}M\ \forall j\geq 0$

$\Rightarrow \forall \ell\geq 1: P^{\ell m}\cdot G_j M \subset P^{\ell m}\cdot F_{j+j_0}M$

$$\overset{(*)\text{ for } P^\ell}{\subset} F_{\ell m k+j+j_0-\ell}M \subset G_{\ell m k+j+2j_0-\ell}M$$

For $\ell = 2j_0+1$ and $m' = \ell m$, we get

$$p^{m'} \cdot G_j M \subset G_{m'k+j-1} M \quad \forall j \geq 0$$

$$\Rightarrow p \in \sqrt{I(M, G_\cdot M)}$$

by symmetrie $\sqrt{I(M,G_\cdot M)} \subset \sqrt{I(M,F_\cdot M)}$

$\square$

<u>Def</u>: Let M be a f.g. left A_n-module, then $\mathrm{char}(M) \subset K^{2n}$ is the closed alg. set defined by $J(M)$, and is called characteristic variety of M.

Define $d^{\deg}(M) := \dim \mathrm{char}(M) =$

$$\dim\left(gr^{F^{\deg}} A_n / J(M)\right)$$

to be its dimension

Example: Consider M f.g. left A_n-mod, then we have:

$$M \text{ is f.g. over } K[x_1, \dots, x_n] \iff \begin{array}{l} \mathrm{Char}(M) = \text{zero section of proj.} \\ K^{2n} \longrightarrow K^n \\ (\underline{x}, \xi) \longmapsto \underline{x} \end{array}$$

Proof: "$\Longrightarrow$": put $F_{-1}M = \{0\}$, $F_k M = M \ \forall k \geq 0$ is good filtration (w.r.t. $F^{ord} A_n$).

Clearly, $gr_k^F M = 0 \ \forall k \neq 0 \Rightarrow K[P] \in gr_j^{F^{ord}} A_n$, $j > 0$

$\Rightarrow [P] \cdot gr_\bullet^F M = 0 \Rightarrow \underline{I}(M, F_\bullet M) = (\xi_1, \dots, \xi_n)$

"$\Longleftarrow$": choose good filtration $F_\bullet M$, then $gr_\bullet^F M$ is $gr_\bullet^{F^{ord}} A_n$-finitely generated. Assumpt: $\mathrm{char}(M) = \text{zero section of } (K^{2n} \to K^n)$, hence:

$\forall i = 1 \dots n : \exists k_i \geq 1 : \xi_i^{k_i} \in \underline{I}(M, F_\bullet M)$

hence $(\xi_1^{e_1} \cdot \ldots \cdot \xi_n^{e_n}) \, gr^F_\cdot M = 0 \quad \forall e_1 + \ldots + e_n >> 0$

$\Rightarrow gr^F_\cdot M = K[\underline{x}] \cdot \langle \xi_1^{e_1} \cdot \ldots \cdot \xi_n^{e_n} \cdot [m] \mid$ finitely many e_i

$\langle [m] \rangle_{gr^F_\cdot A_n} = gr^F M \rangle$

$\Rightarrow M = K[\underline{x}] \cdot \langle \partial_{x_1}^{e_1} \cdot \ldots \cdot \partial_{x_n}^{e_n} m \mid \ldots\ldots \rangle$

$\square$

<u>Problem</u>: for M f.g. left A_n-module, we have defined $d(M)$ (its dimension using good filtration compatible with $F^B A_n$) and $d^{deg}(M)$ ($= \dim(char(M))$ using good filtration compatible with $F^{ord} A_n$)

<u>Aim</u>: show $d(M) = d^{deg}(M)$

in particular, this yields $d^{deg}(M) = n$, so we see that $\dim(\mathrm{char}(M)) \subset [n, 2n]$ (to be re-proven later)

Idea of prove: study more generally (non-comm.) ring R, with $1 \in R$, equipped with filtration

$$\{0\} = F_{-1}R \subset F_0 R \subset F_1 R \subset \dots$$

s.t. $1 \in F_0 R$, $F_k R \cdot F_\ell R \subset F_{k+\ell} R$ and such that $S := gr^F R$ is (a) commutative + noetherian
(b) regular of $\dim = 2n$

ex: 1.) $R = A_n$, $F_\cdot = F_\cdot^B A_n$
2.) $R = A_n$, $F_\cdot = F_\cdot^{ord} A_n$

reminder about Ext-groups: let M, N be left R-modules $\Rightarrow$ $\mathrm{Hom}_R(M, N) = \{f : M \to N$ left R-linear$\}$

$\mathrm{Ext}^k_R(M,N)$: j-th derived functor of $\mathrm{Hom}_R(-,N)$, i.e.,

choose resolution of M by free R-mod's

$$\cdots \to L_1 \to L_0 \to M \to 0$$

$\Rightarrow \mathrm{Ext}^k_R(M,N) :=$

$$H^k\left[0 \to \mathrm{Hom}_R(L_0,N) \to \mathrm{Hom}_R(L_1,N) \to \mathrm{Hom}_R(L_2,N) \to \cdots\right]$$

<u>Fact</u>: R non-commutative $\Rightarrow$ $\mathrm{Hom}_R(M,N)$ is neither left nor right R-mod.

<u>But</u> (exercise): If $N=R$ (seen as right R-module), then $\mathrm{Hom}_R(M,R)$ (as well as all $\mathrm{Ext}^k_R(M,R)$) are right R-mod's

Similarly, if M is right module then $\mathrm{Ext}^k_R(M,R)$ are naturally left modules

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Let now M be a f.g. R-module.

As for $R = A_n$, a filtration $F_\bullet M$ is called compatible with $F_\bullet R$ if $F_k R \cdot F_e M \subset F_{k+e} M$, and it is called good if $\mathrm{gr}^F_\bullet M$ is f.g. over $S = \mathrm{gr}^F_\bullet R$. Similarly, M f.g. over $R \iff$

$\exists F_\bullet M$ good filtration

again, we have $J(M) := \sqrt{\mathrm{Ann}_S(\mathrm{gr}^F_\bullet M)} \subset S$, independent of choice of $F_\bullet M$, and we put $\mathrm{char}(M) = V(J(M)) \subset \mathrm{Spec}(S)$

(more precisely: $\mathrm{gr}^F_\bullet M$ defines coherent sheaf on $\mathrm{Spec}(S)$ with support given by $\mathrm{char}(M) = V(J(M))$)

Then we have the following major result:

<u>Theorem</u>: Put

$d(M) := \dim \mathrm{char}(M) = \dim^{\mathrm{Krull}} S/\mathfrak{J}(M)$

(dimension)

$j(M) := \min \{ k \geq 0 \mid \mathrm{Ext}_R^k(M,R) \neq 0 \}$

(grade number)

Then we have: $d(M) + j(M) = \dim S$

In particular, since $j(M)$ does not depend on choice of $F_\bullet R$, neither does $d(M)$.

Several steps towards the proof: Consider first the case where $R := S$ is commutative.

<u>Prop</u>: Let S be a comm, reg. noetherian of dim $=2n$, let M be f.g. S-module, put

$J(M) := \sqrt{\operatorname{ann}_S(M)} \subset S$ and $d(M) = \dim S/J(M)$

$j(M) = \min\{k \geq 0 \mid \operatorname{Ext}^k_S(M,S) \neq 0\}$, then

(a) $\forall j \in \{0,\dots,2n\}$: $\operatorname{Ext}^j_S(M,S) \neq 0 \Rightarrow j \in [2n-d(M), 2n]$

(i.e. $j(M) \geq 2n - d(M)$)

(b) $\forall j \in \{0,\dots,2n\}$: $d(\operatorname{Ext}^j_S(M,S)) \leq 2n-j$

(c) $d(\operatorname{Ext}^{j(M)}_S(M,S)) = d(M)$

(d) $j(M) + d(M) = 2n$

<u>Idea of proof</u>: (a)+(b) reduce to case S is local:

$$\operatorname{Ext}^j_S(M,S) = 0 \iff S_m \otimes \operatorname{Ext}^j_S(M,S) = \operatorname{Ext}^j_{S_m}(M \otimes S_m, S_m) = 0$$

$$\forall m \supset J(M)$$

so assume S is reg. local ring of dim $= 2n$

show (a)+(b) by induction over $d(M)$:

$$d(M)=0 \overset{S \text{ local}}{\Longrightarrow} \sqrt{J(M)} = \mathfrak{m} \underset{\text{NAK}}{\overset{M \text{ f.g.}}{\Longrightarrow}} \exists \ell : \mathfrak{m}^\ell M = 0$$

reduce to case $\ell = 1$ by chain $M \supset \mathfrak{m}M \supset \ldots \supset \mathfrak{m}^\ell M = \{0\}$

and $0 \to \mathfrak{m}^k M \to M \to M/\mathfrak{m}^k M \to 0$ and

long exact sequence of Ext-modules.

reduce further to case $M \cong S/\mathfrak{m}$ and to

case where $S/\mathfrak{m} = \kappa(\mathfrak{m})$

Then $S/\mathfrak{m}$ is Gorenstein, and

$$\mathrm{Ext}^j_S(S/\mathfrak{m}, S) = \begin{cases} 0 & \forall j < 2n \\ S/\mathfrak{m} & j = 2n \end{cases}$$

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now supp. $d(M) > 0$, take $f \in \mathfrak{m}$, f not 0-div. on M

$$\Rightarrow \quad 0 \to M \xrightarrow{f} M \to M/fM \to 0 \quad (*)$$

$$d(M/fM) = d(M) - 1$$

put $E^j := \operatorname{Ext}^j_S(M,S)$, $F^j := \operatorname{Ext}^j_S(M/fM,S)$

induction hypothesis: (a) + (b) hold for mod's of dim < $d(M)$,

therefore: (a) $F^j \neq 0 \Rightarrow j \in [2n - d(M/fM), 2n]$

$$= [2n - d(M)+1, 2n]$$

(b) $d(F^j) \leq 2n - j$

long exact Ext-sequence of (*):

$$\to F^j \to E^j \xrightarrow{f} E^j \to F^{j+1} \to \qquad (*d)$$

$F^j = F^{j+1} = 0 \quad \forall j < 2n - d(M) \Rightarrow f : E^j \to E^j$ iso

$\overset{\text{NAK}}{\rightsquigarrow} E^j = 0$ & $E^j / f \cdot E^j \subset F^{j+1}$

$\Rightarrow 2n - (j+1) \geq d(F^{j+1}) \geq d\left(E^j / f \cdot E^j\right) \geq d(E^j) - 1$

$\to d(E^j) \leq 2n - j$

(c) from (a) $j(M) \geq 2n - d(M)$

(b) $\forall j: d(E^j) \leq 2n - j \underset{\substack{j \geq j(M) \\ \text{or else} \\ d(E^j)=0}}{\leq} 2n - j(M)$

$\leq d(M)$ (with $d(E^j) < d(M)$ if $j > j(M)$)

to show: $d(E^{j(M)}) = d(M)$

supp: $d(E^{j(M)}) < d(M)$

hence $d(E^j) < d(M)$ $\forall j \geq 0$, define

$E := \bigoplus_{j=2n-d} E^j \Rightarrow d(E) < d(M)$, hence $J(E) \supsetneq J(M)$

$\exists g \in J(E) \setminus J(M)$, replace M by M_g

$\Rightarrow \mathrm{Ext}^j(M, S) = 0$ $\forall j \geq 0$ if M is f.g.

(d) from (α): $j(M) \geq 2n - d(M)$

$$d(M) \overset{(c)}{=} d(E^{j(M)}) \overset{(\beta)}{\leq} 2n - j(M)$$

$$\Rightarrow j(M) + d(M) = 2n$$

<u>remains</u>: R as in theorem

M f.g. left R-mod, F. M good

$$\Rightarrow \quad j(M) = j(gr^F M)$$
