Reminder from last time:

R (possibly non-commutative) ring, $F_\bullet R$, $1 \in F_0 R$

$F_k R \cdot F_\ell R \subset F_{k+\ell} R$, $S = \mathrm{gr}^F_\bullet R$ comm, reg, noeth,

$\dim S = 2n$

M f.g left R-mod $\rightsquigarrow$ chose good filtration $F_\bullet M$

$\mathrm{char}(M) = V\left(\sqrt{\mathrm{ann}_S(\mathrm{gr}^F M)}\right) \subset \mathrm{Spec}(S)$,

$d(M) := \dim \mathrm{char}(M)$ (does not depend on choice of $F_\bullet M$, but a priori on choice of $F_\bullet R$)

$j(M) = \min_k \{ \mathrm{Ext}^k_R(M, R) \neq 0 \}$

Then $\quad j(M) + d(M) = 2n$

<u>Consequence</u> $d(M)$ can be computed with any choice of $F_\bullet R$, in particular, for

$R = A_n$, $F = \begin{cases} F^{\mathrm{ord}} \\ F^B \end{cases}$ $\quad d^{\deg}(M) = d^{\mathrm{Bernstein}}(M)$

we have shown: If $R = S$ is commutative

then $d(M) + j(M) = 2n$ holds

where $d(M) = \dim(S/\mathrm{ann}_S M)$

$j(M) = \min_k \{ \mathrm{Ext}^k_S(M,S) \neq 0 \}$

Now let R be arbitrary, $F_\bullet R$ as above, M left R-mod

$F_\bullet M$ good filtration, then since by definition

$d(M) = d(\mathrm{gr}^F_\bullet M)$, it remains to show

$j(M) = j(\mathrm{gr}^F_\bullet M)$, hence, to study relation

between $\mathrm{Ext}^k_R(M,R)$ and $\mathrm{Ext}^k_S(\mathrm{gr}^F_\bullet M, S)$

Preliminary considerations:

<u>Prop</u>: $\exists$ resolution of M by free R-mod's

$$\cdots \to L_2 \to L_1 \to L_0 \to M \to 0$$

where each L_i carries good filtration $F_\bullet L_i$,

where $d^i : F_k L_i \to F_k L_{i-1}$ and where:

1.) $gr^F L_i$ is S-free with $rk_S(gr^F L_i) = rk_R L_i$

2.) the complex of S-modules

$$\dots \longrightarrow gr^F L_2 \to gr^F L_1 \to gr^F L_0 \to gr^F M \to 0$$

is exact

Proof: for $e \in \mathbb{Z}$, put $R(e) := R$ with filtration $F_k R(e) := F_{k+e} R$. Now we have

$gr^F M = \sum_{j=1}^{r} S[m_j]$ with $[m_j] \in gr^F_{e_j} M$, i.e.

$gr^F_k M = \sum_{j=1}^{r} S_{k-e_j}[m_j] \Rightarrow F_k M = \sum_{j=1}^{r} F_{k-e_j} R \cdot m_j$

Put $L_0 := \bigoplus_{j=1}^{r} R(-e_j)$ then $L_0 \twoheadrightarrow M$

s.t. $\mathrm{gr}^F L_0 \twoheadrightarrow \mathrm{gr}^F M$. Put $M' := \ker(L_0 \twoheadrightarrow M)$,

$\Rightarrow \quad 0 \to \mathrm{gr}^F M' \to \mathrm{gr}^F L \to \mathrm{gr}^F M \to 0$

$\xRightarrow{S \text{ noeth}}$ $\mathrm{gr}^F M'$ is f.g. over S $\Rightarrow$ $F_\bullet M'$ good filtration

apply same argument to $(M', F_\bullet M')$ to construct

$L_1 \twoheadrightarrow M'$, i.e. $L_1 \to L_0 \to M \to 0$

etc. (Th: stops after $\le \dim M \le 2n$ steps) $\square$

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given free resolution

$$\ldots \to L_1 \to L_0 \to M \to 0$$

as above, put $L_i^* := \mathrm{Hom}_R(L_i, R)$ (right R-module) $\rightsquigarrow$ $\mathrm{Ext}^i_R(M,R) = H^i(0 \to L_0^* \to L_1^* \to \ldots)$

<u>Def</u>: Let $(L, F_\bullet L)$ be f.g. R-mod with good filtr.

Put. $F_k L^* := \{\phi \in L^* \mid \phi(F_j L) \subset F_{j+k} R \quad \forall j \ge 0\}$

<u>Lemma</u>: L^* is f.g over R, and $F_\bullet L^*$ is a good filtration

<u>Proof</u>: exercise, use $gr_\bullet^F L^* \hookrightarrow Hom_S(gr_\bullet^F L, S)$

for any f.g. R-module M

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actually, for L free R-mod, we have

$$gr_\bullet^F L^* \simeq Hom_S(gr_\bullet^F L, S)$$

and so

$$Ext_R^k(M,R) = H^k\left[0 \to L_0^* \to L_1^* \to \dots\right]$$

$$Ext_S^k(gr^F M, R) = H^k\left[0 \to gr_\bullet^F L_0^* \to gr_\bullet^F L_1^* \to \dots\right]$$

hence, need:

$$H^k\left[0 \to L_0^* \to L_1^* \to \dots\right] = 0 \iff H^k\left[0 \to gr_\bullet^F L_0^* \to gr_\bullet^F L_1^* \to \dots\right] = 0$$

Tool: Spectral sequence of filtered complex

let complex $\cdots \to K^{n-1} \xrightarrow{d} K^n \xrightarrow{d} K^{n+1} \to \cdots$

with filtration $F_\bullet K^\bullet$ be given (i.e. $d(F_k K^i) \subset F_k K^{i+1}$), then $\exists$ spectral

sequence $E_0^{p,q} := gr_p^F K^{p+q}$; $E_1^{p,q} := H^{p+q}(gr_p^F K^\bullet)$

$$\Longrightarrow E_\infty^{p,q} := gr_p^F H^{p+q}(K^\bullet)$$

notice:

$$H^{p+q}(gr_p^F K^\bullet) \simeq \frac{F_p K^{p+q} \cap d^{-1}(F_{p-1} K^{p+q-1})}{F_{p-1} K^{p+q} + d(F_p K^{p+q-1})}$$

and

$$gr_p^F H^{p+q}(K^\bullet) \simeq \frac{F_p K^{p+q} \cap \ker d}{F_{p-1} K^{p+q} \cap \ker d + F_p K^{p+q} \cap d(K^{p+q-1})}$$

in general $\neq$

applies to case where $K^n := L_n^*$, dual of free resolution $\to L_1 \to L_0 \to M \to 0$ for $(M, F_\bullet M)$ well-filtered left R-mod. We have induced good filtration $F_\bullet L_n^*$, and we consider spectral sequence of filtered complex $(L_\bullet^*, F_\bullet)$. Then

<u>Prop</u>: (a) $\forall n : \exists p_0 : F_p L_{n+1}^* \cap d(L_n^*) = F_p L_{n+1}^* \cap d(F_{p+p_0} L_n^*)$

(b) $E_k^{p,q}$ converges to $E_\infty^{p,q}$ (i.e. $\exists p_0 : E_{p_0}^{p,q} = E_\infty^{p,q}$)

(c) $j(M) = j(gr^F M) =: j_0$

(d) $d(\underbrace{Ext^j(M,R)}_{\text{right } R\text{-mod}}) \leq 2n - j \quad \forall j \geq 0$

(e) $d(Ext^{j(M)}(M,R)) = 2n - j(M)$

(f) $j(M) + d(M) = 2n$

<u>Proof (some ideas):</u>

(a) <u>z.z:</u> $F_p L^\bullet_{n+1} \cap \tilde{d}(L^\bullet_n) \subset F_p L^\bullet_{n+1} \cap \tilde{d}(F_{p+p_0} L^\bullet_n)$

$\tilde{d} L^\bullet_n \subset L^\bullet_{n+1}$ submodule with good filtrations

$G_\bullet \tilde{d} L^\bullet_n := \tilde{d} L^\bullet_n \cap F_\bullet L^\bullet_{n+1}$

$G'_\bullet \tilde{d} L^\bullet_n := \tilde{d}(F_\bullet L^\bullet_n)$

$$\exists p_0: \underset{\parallel}{G_p \tilde{d} L^\bullet_n} \subset \underset{\parallel}{G'_{p+p_0} \tilde{d} L^\bullet}$$

$$\tilde{d} L^\bullet_n \cap F_p L^\bullet_{n+1} \subset \tilde{d}(F_{p+p_0} L^\bullet_n) \quad \Big| \cap F_p L^\bullet_{n+1}$$

$$F_p L^\bullet_{n+1} \cap \tilde{d} L^\bullet_n \subset F_p L^\bullet_{n+1} \cap \tilde{d}(F_{p+p_0} L^\bullet_n)$$

(b) general convergence criterion spectral sequences of filtered complexes, using (a)

(c) By definition: $Ext^j_S(gr^F_\bullet M, S) = 0 \quad \forall j < j_0$

but $Ext^j_S(gr^F M, S) \simeq H^j(gr^F_\bullet L^\bullet_\bullet) =$

$$\bigoplus_p H^{p+(j-p)}\left(gr_p^F L^*_\bullet\right) \simeq \bigoplus_p E_1^{p,j-p}$$

$$\Rightarrow E_k^{p,j-p} = 0 \quad \forall j < j_0 \left(= j\left(gr^F M\right)\right), \forall k \geq 1$$

since $E_k^{p,j-p} = E_\infty^{p,j-p} \quad \forall k \gg 0$

$$\Rightarrow gr^F_\bullet Ext^j_R(M,R) = 0 \quad \forall j < \overbrace{j\left(gr^F M\right)}^{=j_0}$$

$$\Rightarrow Ext^j_R(M,R) = 0 \quad \forall j < j_0$$

$$\Rightarrow j(M) \geq j_0$$

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recall from last time that $d\left(Ext^j_S\left(gr^F M, S\right)\right) \leq 2n - j$ $\forall j \geq 0$ where $d(-) := \dim\left(supp(-)\right)$

$$\Rightarrow d\left(E_1^{p,j-p}\right) \leq 2n - j \quad \forall j \geq 0$$

$$\Rightarrow \underbrace{d\left(gr^F Ext^j_R(M,R)\right)}_{\overset{\text{by def}}{=}} \leq 2n - j$$

$$d\left(Ext^j_R(M,R)\right) \leq 2n - j \qquad (d)$$

by definition, $Ext^{j_0-1}_S(gr^F M, S) = 0 \Rightarrow E_1^{p,j_0-p-1} = 0 \quad \forall p$

$\Rightarrow E_2^{p,j_0-p} = \ker\left(\partial : E_1^{p,j_0-p} \to E_1^{p,j_0+1-p}\right)$

but we also have:

- $d\left(Ext_S^{j_0+1}(gr^F M, S)\right) \le 2n - j_0 - 1 \Rightarrow d\left(E^{p,j_0+1-p}\right) \le 2n - j_0 - 1$
- $d\left(Ext_S^{j_0}(gr^F M, S)\right) \overset{!}{=} 2n - j_0 \Rightarrow d\left(E^{p,j_0-p}\right) = 2n - j_0$

Hence: $d\left(E_2^{p,j_0-p}\right) = 2n - j_0$

by induction $d\left(E_k^{p,j_0-p}\right) = 2n - j_0$

$\Rightarrow d\left(E_\infty^{p,j_0-p}\right) = 2n - j_0 \Rightarrow d\left(Ext_R^{j_0}(M,R)\right) = 2n - j_0$,

hence $\boxed{Ext_R^{j_0}(M,R) \neq 0}$, and therefore

$j(M) \le j_0 = j(gr^F M) \Rightarrow j(M) = j(gr^F M)$ (c)

since we had shown last time $j_0 + \underset{\substack{\| \\ d(gr^F M)}}{d(M)} = 2n$

it follows that $j(M) + d(M) = 2n$ $\square$