Recall from last time: R n.c. ring,

$F.R$ s.t. $S = \mathrm{gr}^F R$ is comm, reg., noeth., dim $= 2n$.

M left or right f.g. R-mod. Then

1.) $j(M) = j(\mathrm{gr}^F M)$ $\left(j(-) = \min_k \mathrm{Ext}^k_?(-,?) \neq 0 \right)$

2.) $d(\mathrm{Ext}^j_R(M,R)) \leq 2n - j$ $\quad \forall j \geq 0$

3.) $d(\mathrm{Ext}^{j(M)}_R(M,R)) = 2n - j(M)$

4.) $d(M) + j(M) = 2n$ (follows from 1.) and $d(\mathrm{gr}^F M) + j(\mathrm{gr}^F M) = 2n$)

For $R = A_n$, $\exists$ Bernstein's inequality:

$\forall$ f.g. left/right A_n-mod $M \neq 0$, we have $d(M) \geq n$ $\quad (*)$

Suppose more generally that R is a ring where $(*)$ holds true

$\Longleftrightarrow \forall M \neq 0: j(M) \leq n \Longleftrightarrow \exists k \leq n: \mathrm{Ext}^k_R(M,R) \neq 0$

Corollary: Assume $(*)$. Let M be f.g left/right R-mod, $M \neq 0$. Then $\mathrm{Ext}^k_R(M,R) = 0$ $\forall k > n$.

Pf: M f.g left/right R-mod $\Longrightarrow \mathrm{Ext}^k_R(M,R)$ is f.g right/left R-mod

From 2.) above: $d(\mathrm{Ext}^k_R(M,R)) \leq 2n-k$

but by (*), we have Bernstein's inequality for $\mathrm{Ext}^k_R(M,R)$, i.e. $d(\mathrm{Ext}^k_R(M,R)) \geq n$, if $\mathrm{Ext}^k_R(M,R) \neq 0$. Hence, by red equations, if $k > n \Rightarrow \mathrm{Ext}^k_R(M,R) = 0$ $\square$

as for $R = A_n$, call M holonomic iff $M = 0$ or $d(M) = n$. Then

<u>Corollary</u>: M f.g. R-mod is holonomic $\Longleftrightarrow \mathrm{Ext}^k_R(M,R) = 0 \quad \forall k \neq n$

<u>Proof</u>: clear if $M = 0$. Let $M \neq 0$. We have $j(M) + d(M) = 2n$, hence: M holonomic $\Longleftrightarrow$ $j(M) = n \overset{\text{Corollary above}}{\Longleftrightarrow} \mathrm{Ext}^k_R(M,R) = 0 \;\; \forall k \neq n$ $\square$

<u>Def</u>: M f.g. left / right holon. R-mod then $M^* := \mathrm{Ext}^n_R(M,R)$ is holonomic

dual of M. (For $R = A_n$, put $\mathbb{D}M :=$

$(M^*)^\vee$ ← left module associated to M^* by transposition $P \cdot m := m \cdot P^{tr}$)

<u>Lemma:</u> M holonomic $\Rightarrow M^*$ holonomic

<u>Pf:</u> $d(M^*) = d(\mathrm{Ext}^n_R(M,R)) \overset{2.)}{=} 2n - n = n$ □

<u>Fact</u> (exercise) $\phi : \mathrm{Hol}^{left}(R) \longrightarrow \mathrm{Hol}^{right}(R)^{op}$

$$M \longmapsto M^*$$

is contravariant functor

<u>Proposition:</u> (a) $(M^*)^* \simeq M$

(b) ϕ is an equivalence of categories

<u>Proof:</u> (b) follows from (a), hence, show (a)

Let M be holonomic, consider free resolution

$$\dots \to L_1 \to L_0 \to M \to 0$$

dual complex:

$$0 \to L_0^* \to L_1^* \to \dots \to L_{n-1}^* \to L_n^* \to L_{n+1}^* \to \dots$$

we have $H^k(\text{-- " --}) = \begin{cases} M^* = \mathrm{Ext}_R^n(M,R) & \text{for } k=n \\ 0 & \text{else} \end{cases}$

consider also free resolution of M^*:

$$\dots \to K_1 \to K_0 \to M^* \to 0$$

<u>Claim</u>: $\exists$ quasi-isomorphism

$$\begin{tikzcd} \dots \arrow[r] & K_1 \arrow[r] \arrow[d] & K_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \dots \\ \dots \arrow[r] & L_{n-1}^* \arrow[r, "d^{n-1}"] & L_n^* \arrow[r, "d^n"] & L_{n+1}^* \arrow[r] & \dots \end{tikzcd}$$

<u>Proof of claim</u>: follows from abstract nonsense,

concrete construction: Consider $\ker(d^n) \subset L_n^*$, then

$M^* \simeq \ker d^n / \operatorname{Im} d^{n-1}$. Hence $\exists$ surjection $\ker(d^n) \twoheadrightarrow M^*$.

By construction, $\exists$ surjection $K_0 \twoheadrightarrow M^*$

K_0 is R-free $\rightsquigarrow$ $\exists$ lifting:

$$\begin{tikzcd} & K_0 \arrow[dl, dashed] \arrow[d] \\ \ker d^n \arrow[r, two heads] & M^* \end{tikzcd}$$

define $f_0 : K_0 \to \ker d^n \hookrightarrow L^*_n$. Hence

$d^n \circ f_0 = 0$, so:

$$\begin{tikzcd} \cdots \arrow[r] & K_1 \arrow[r] \arrow[dr, red, "\alpha"'] & K_0 \arrow[r] \arrow[d, "f_0"] \arrow[dr, phantom, "\circlearrowleft"] & 0 \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & L^*_{n-1} \arrow[r, "d^{n-1}"'] & L^*_n \arrow[r, "d^n"'] & L^*_{n+1} \arrow[r] & \cdots \end{tikzcd}$$

we have $d^n \circ \alpha = 0 \Rightarrow$ have map $K_1 \to \ker d^n / \operatorname{Im} d^{n-1}$
$\simeq M^*$ but this map is zero, since $M^* \simeq \operatorname{coker}(K_1 \to K_0)$

Hence: $\operatorname{Im}(\alpha) \subset \operatorname{Im}(d^{n-1})$, K_1 free $\Rightarrow \exists$ lift

$$\begin{tikzcd} \cdots \arrow[r] & K_1 \arrow[r] \arrow[d, "f_1"] & K_0 \arrow[r] \arrow[d, "f_0"] & 0 \arrow[r] & \cdots \\ \cdots \arrow[r] & L^*_{n-1} \arrow[r] & L^*_n \arrow[r] & L^*_{n+1} \arrow[r] & {} \end{tikzcd}$$

etc.

$\square$ Pf. of claim

Apply $\mathrm{Hom}_R(-,R)$ to morphism of complexes $K_\bullet \to (L_\bullet)^*[-n]$.

Since $((L_k^*))^* \simeq L_k$ (free R-modules), we get morphism of complexes:

$$\begin{array}{ccccccccccc}
\cdots & \to & L_{n+1} & \to & L_n & \to & L_{n-1} & \to & \cdots \\
& \to & \downarrow 0 & & \downarrow f_0^* & & \downarrow f_1^* & & \downarrow \\
\cdots & \to & 0 & \to & K_0^* & \to & K_1^* & \to & K_2^* & \to & \cdots
\end{array}$$

exercise: This is still a quasi-isomorphism. We have $H_k(L_\bullet) = \begin{cases} M & k=0 \\ 0 & \text{else} \end{cases}$. On the other hand, M^* is holonomic (Lemma above), hence

$$H^k(K_\bullet^*) = \begin{cases} (M^*)^* = \mathrm{Ext}^n_R(M^*, R) & k=n \\ 0 & \text{else.} \end{cases}$$

Hence, the chain morphism $f_\bullet^*$ induces isomorphism $M \simeq (M^*)^*$ $\square$

Corollary: M holonomic $\Rightarrow \mathrm{char}(M) = \mathrm{char}(M^*)$

Pf.: Choose good filtration $F_\bullet M$, then

$$\mathrm{char}(M) = \mathrm{Supp}(\mathrm{gr}^F_\bullet M) = \mathrm{Spec}\left(\sqrt{\mathrm{ann}_{\substack{S \\ \| \\ \mathrm{gr}^F R}} \mathrm{gr}^F(M)}\right)$$

we obtain induced filtration $F_\bullet M^* = F_\bullet \mathrm{Ext}^n_R(M,R)$

[from filtered free resolution $(L_\bullet, F_\bullet) \xrightarrow{\sim} (M, F_\bullet M)$]

We had convoluted spectral sequence

$$E_1^{p,q} := H^{p+q}(\mathrm{gr}^F_p L^*_\bullet) \Rightarrow E_\infty^{p,q} = \mathrm{gr}^F_p \mathrm{Ext}^{p+q}_R(M,R)$$

Convergence implies that $E_\infty^{p,q}$ is subquotient of $E_1^{p,q}$

$$\Rightarrow \mathrm{supp}(E_\infty^{p,q}) \subset \mathrm{supp}(E_1^{p,q})$$

$$\Rightarrow \mathrm{supp}\Big(\overbrace{\mathrm{gr}^F_\bullet \mathrm{Ext}^n_R(M,R)}^{\oplus_p E_\infty^{p,n-p}}\Big) \subset \mathrm{supp}\Big(\overbrace{\mathrm{Ext}^n_S(\mathrm{gr}^F_\bullet M, S)}^{\oplus_p E_1^{p,n-p}}\Big)$$

Clearly: $\mathrm{supp}(\mathrm{Ext}^n_S(\mathrm{gr}^F M, S)) \subset \mathrm{supp}(\mathrm{gr}^F M) = \mathrm{char}(M)$

It follows that

$$\mathrm{char}(M^*) = \mathrm{supp}\left(\mathrm{gr}^F_\bullet \mathrm{Ext}^n_R(M,R)\right) \subset \mathrm{char}(M)$$

From $(M^*)^* \simeq M$, we also conclude that $\mathrm{char}(M) \subset \mathrm{char}(M^*)$ $\square$

<u>Remarks</u>: (1) We have shown that for $R = A_n$ $\forall$ f.g. R-modules M, ascending chains

$$0 \subseteq N_1 \subseteq N_2 \subseteq \ldots \subseteq M$$

become stationary (length $\leq m(M)$).

also seen: M holonomic $\Longrightarrow$ all descending chains $M \supseteq M_1 \supseteq M_2 \supseteq \ldots$ become stationary

This can be reproved using the holonomic dual: $\mathrm{Ext}^n_R(-,R)$ is contravariant,

hence $M \supseteq M_1 \supseteq M_2 \supseteq \ldots\ldots$ becomes

ascending chain for M^* which is stationary,

hence, also $M \supseteq M_1 \supseteq M_2 \supseteq \ldots\ldots$ is

stationary by $((-)^*)^* = id_{Hol(R)}$ for

rings R s.t. Bernstein's inequality holds

2.) example for computation of holonomic dual:

let $R = A_1$, $M = A_1/A_1 \cdot P$ for operator $P \in A_1$.

Free resolution: $L_\bullet = [0 \to \underset{L_1}{A_1} \xrightarrow{\cdot P} \underset{L_0}{A_1} \to 0]$

$$\rightsquigarrow \quad 0 \to L_1 \to L_0 \to M \to 0 \quad \Big| \ \mathrm{Hom}_{A_1}(L_\bullet, A_1)$$

$$0 \to A_1 \cong \mathrm{Hom}_{A_1}(A_1, A_1) \xrightarrow{P\cdot} \mathrm{Hom}_{A_1}(A_1, A_1) \cong A_1 \to 0$$

$\Rightarrow H^0(L^*_\bullet) = 0$ (hence, putting $\mathrm{Hom}_{A_1}(M, A_1)$ for the dual module makes no sense)

and $H^1(L^a_\bullet) = \mathrm{Ext}_{A_1}(M, A_1) = A_1 / P \cdot A_1$

also $\mathbb{D}M = [M^a]^{\text{left}} \simeq A_1 / A_1 \cdot P^{tr}$