Next aim: Introduce and study sheaf of alg. differential operators on algebraic varieties

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Preliminaries / Reminder:

Let K field ($\operatorname{char} K = 0$) and X an alg. variety over K (integral, separated scheme, locally of finite type), later: X regular, i.e. $\dim \mathcal{O}_{X,x} = \dim_{\mathcal{O}_{X,x}/m_x}(m_x/m_x^2)$

Kähler differentials: $\forall$ K-algebras A, we have $\Omega^1_{A/K}$ s.t. $\operatorname{Der}_K(A, M) \simeq \operatorname{Hom}(\Omega^1_{A/K}, M)$

Globally: $\Omega^1_{X/K}$ coherent $\mathcal{O}_X$-module s.t.

$\forall\, U \subset X$ open: $\Gamma(U, \Omega^1_{X/K}) := \Omega^1_{A/K}$, $A = \Gamma(U, \mathcal{O}_X)$

$\exists$ universal K-linear derivation $d: \mathcal{O}_X \to \Omega^1_{X/K}$

tangent sheaf $\Theta_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X)$

X non-singular / regular $\Leftrightarrow \Omega^1_{X/k}$ locally free of rk $= \dim X$

<u>local coordinates</u>: $x \in X$ closed point

$\rightsquigarrow \exists\, U \subset X, x \in U$ affine open, s.t. $\exists\, x_1, \dots, x_n \in \Gamma(U, \mathcal{O}_X)$

$$(\Omega^1_{X/k})_{|U} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{X|U} \cdot dx_i$$

$$\left(\text{and } \Theta_{X|U} \cong \bigoplus \mathcal{O}_{X|U} \cdot \partial_{x_i}\right)$$

$(x_1, \dots, x_n) : U \to \mathbb{A}^n$ étale

Differential operators: start with affine case

$A := \Gamma(U, \mathcal{O}_X)$. Aim: define subalgebra

$D(A) \subset \mathrm{End}_k(A)$, with filtration $F_\bullet D(A)$.

<u>Rk</u>: $\forall f \in A \rightsquigarrow f \cdot - \in \mathrm{End}_k(A)$, moreover

$\forall P \in \mathrm{End}_k(A)$: $P = P(1)\cdot$ (equality in $\mathrm{End}_k(A)$)

$$\iff P \in \mathrm{End}_A(A) \quad P(f) = P(1)\cdot f$$

$$\iff [P,f] = 0 \quad \forall f \in A$$

Hence, we put (Grothendieck)

$$F_0 D(A) := \{P \in \mathrm{End}_k(A) \mid [P,f] = 0 \ \forall f \in A\} \simeq A$$

$$P \longmapsto P(1)$$

recursively: $F_k D(A) := \{P \in \mathrm{End}_k(A) \mid [P,f] \in F_{k-1} D(A) \ \forall f \in A\}$

example: $\delta \in \mathrm{Der}_k(A) \Rightarrow \forall f,g \in A$:

$$[\delta, f](g) = \delta(fg) - f\delta(g) = \delta(f)\cdot g \in A = F_0 D(A)$$

$$\Rightarrow \delta \in F_1 D(A)$$

on the other hand, let $P \in F_1 D(A)$, then $\forall f \in A$:

$\exists p_f \in A$: $[P,f] = p_f$. Hence, $\forall f,g \in A$

$$P(fg) - f\cdot P(g) = [P,f]\cdot g = p_f \cdot g \qquad (*)$$

for $g=1 \Rightarrow P(f) - f P(1) = pf$

$\xrightarrow[\text{ui } (*)]{\text{replace}}$: $P(fg) - f P(g) = g P(f) - fg P(1)$

i.e. $P(fg) - f P(g) - g P(f) + fg P(1) \overset{(**)}{=} 0$

claim: $P - P(1) \in \mathrm{Der}_k(A)$:

$$(P - P(1))(fg) = P(fg) - fg P(1)$$

$$\overset{(**)}{=} f P(g) + g P(f) - 2 \cdot fg P(1)$$

$$= f \big(P(g) - g P(1)\big) + g \big(P(f) - f P(1)\big)$$

$$= f \big(P - P(1)\big)(g) + g \big(P - P(1)\big)(f)$$

Hence: $F_1 D(A) \cong \mathrm{Der}_k(A) \oplus F_0 D(A)$

$$P \longmapsto \big(P - P(1), P(1)\big)$$

Proposition (exercise, similarly to $A = k[x_1, \dots, x_n]$):

A f.g. k-algebra, regular of dim $= n$.

1.) $F_j D(A)$ is f.g. over A

2.) $P \in F_j D(A)$, $Q \in F_i D(A) \Rightarrow P \circ Q \in F_{i+j} D(A)$

$[P, Q] \in F_{i+j-1} D(A)$

3.) $D(A)$ is generated as a subalgebra of $\mathrm{End}_k(A)$ by $\mathrm{Der}_k(A)$ subject to

$[\delta, f] - \delta(f) \quad \forall \delta \in \mathrm{Der}_k(A), f \in A$

4.) $F_j D(A) / F_{j-1} D(A) \simeq \mathrm{Sym}^j \mathrm{Der}_k(A)$

5.) $\mathrm{gr}^F_\bullet D(A) \simeq \mathrm{Sym}_\bullet \mathrm{Der}_k(A)$

(in part., this is comm., reg., noeth., dim $= 2n$ and all the previous chapters apply)

<u>description in local coordinates</u>: $\mathrm{Der}_k(A) := \bigoplus_{i=1}^{n} A \partial_{x_i}$.

$$F_k D(A) = \left\{ P = \sum_{|\alpha| \le k} f_\alpha \cdot \partial^\alpha \;\middle|\; f_\alpha \in A \right\}$$

$$\partial^\alpha := \partial_{x_1}^{\alpha_1} \cdot \ldots \cdot \partial_{x_n}^{\alpha_n}$$

$$D(A) = \bigcup_{k \ge 0} F_k D(A)$$

ex: $A = K[x_1, \ldots, x_n] \Longrightarrow D(A) = A_n(K)$, $F_\bullet D(A) = F^{\mathrm{ord}} A_n(k)$

Prop. (localization): A f.g. k-alg, reg. of dim n

$f \in A \setminus \{0\}$, then

$D(A_f) \simeq A_f \otimes_A D(A)$; $F_j D(A_f) \simeq A_f \otimes_A F_j D(A)$

idea of pf: $P \in D(A_f) \Longrightarrow \exists k \in \mathbb{N}: f^k P \in D(A)$

similarly for $F_j D(A_f)$

Def: $\mathcal{D}_X$ is qcoh sheaf of $\mathcal{O}_X$-modules on X with $\Gamma(U, \mathcal{D}_X) = D(\Gamma(U, \mathcal{O}_X))$ for all $U \subset X$ open. Similarly, $F_j \mathcal{D}_X$ s.t.

$$\Gamma(U, F_j\mathcal{D}_X) = \underbrace{F_j\mathcal{D}(\Gamma(U,\mathcal{O}_X))}_{\text{f.g.}} \quad \Gamma(U,\mathcal{O}_X)-$$

hence $\mathcal{D}_X$ is $\mathcal{O}_X$-qcoh and $F_j\mathcal{D}_X$ is $\mathcal{O}_X$-coherent.

We have $F_0\mathcal{D}_X \simeq \mathcal{O}_X$ and $gr^F_k\mathcal{D}_X \simeq \underline{Sym}^k_{\mathcal{O}_X}(\Theta_X)$

and $gr^F_\bullet\mathcal{D}_X = \bigoplus_k gr^F_k\mathcal{D}_X \simeq \bigoplus_k \underline{Sym}^k_{\mathcal{O}_X}(\Theta_X)$

$\simeq \pi_*\mathcal{O}_{T^*X}$, $\pi : T^*X \to X$

(recall: $T^*X = \underline{Spec}_X(\underline{Sym}^\bullet_{\mathcal{O}_X}(\Theta_X))$)

<u>example</u>: $X = \mathbb{P}^n$. Claim: $\dim_k \Gamma(\mathbb{P}^n, \mathcal{D}_{\mathbb{P}^n}) = \infty$

<u>idea</u>: $v \in \Theta_{\mathbb{P}^n} \leftrightarrow \hat{v} \in \mathbb{A}^{n+1}\setminus\{0\}$

$$\hat{v} = \sum_{i=0}^{n} f_i\partial_{x_i} \quad \deg(f_i) = 1$$

However: $E = \sum x_i\partial_{x_i}$ yields $0 \in \Theta_{\mathbb{P}^n}$

$$\Rightarrow \Gamma(\mathbb{P}^n, \mathcal{D}_{\mathbb{P}^n}) = \frac{\bigoplus k\, x_i\partial_{x_j}}{k\cdot E}$$

<u>exercise</u>: $\Gamma(\mathbb{P}^n, \mathcal{D}_{\mathbb{P}^n}) = \dfrac{K\langle x_i \partial_{x_j}\rangle}{K\langle x_i \partial_{x_j}\rangle \cdot E}$

concretely: $\Gamma(\mathbb{P}^n, \mathcal{D}_{\mathbb{P}^n}) \ni P = \sum_{|\alpha| = |\beta| \leq k} c_\alpha x_0^{\alpha_0} \cdots x_n^{\alpha_n} \partial_0^{\beta_0} \cdots \partial_n^{\beta_n}$

restriction to $\mathbb{A}^n = \{x_0 = 1\}$: $\partial_0 = -(x_1\partial_1 + \ldots + x_n \partial_{x_n})$

<u>Def</u>: X alg. variety, then a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{M}$, together with left / right action by $\mathcal{D}_X$ is called **algebraic left / right $\mathcal{D}_X$-module**

in particular, $\forall\, U \subset X$ affine open, we get $A = \Gamma(U, \mathcal{O}_X)$-mod. $M = \Gamma(U, \mathcal{M})$ with left/right action by $R = \Gamma(U, \mathcal{D}_X) = D(A)$.

Basic examples:

1.) $\mathcal{M} = \mathcal{O}_X$, clear, since $D(\Gamma(U, \mathcal{O}_X))$ acts on $\Gamma(U, \mathcal{O}_X)$

2.) Let $E \to X$ be an algebraic vector bundle and $\mathcal{E}$ its sheaf of sections. A connection on E is a k-lin. morphism

$$\nabla : \mathcal{E} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{E}$$

s.t. $\forall U \subset X$, $\forall f \in \Gamma(U, \mathcal{O}_X)$, $\forall s \in \Gamma(U, \mathcal{E})$ we have:

$$\nabla(f \cdot s) = f \otimes \nabla s + df \otimes s \quad \text{(Leibniz rule)}$$

Rk: $\forall v \in \Gamma(U, \Theta_X)$, we can define composition

$$\nabla_\theta = \iota_\theta \circ \nabla : \mathcal{E} \xrightarrow{\nabla} \Omega^1_X \otimes \mathcal{E} \xrightarrow{\iota_\theta} \mathcal{E}$$

and $\nabla \in \mathcal{H}om_{\mathcal{O}_X}(\Theta_X, \mathcal{E}nd_k(\mathcal{E}))$ (i.e. $\nabla : \Theta_X \times \mathcal{E} \to \mathcal{E}$)

∇ is called flat $\Leftrightarrow [\nabla_\theta, \nabla_\eta] = \nabla_{[\theta, \eta]}$

$\Leftrightarrow \nabla^2 : \mathcal{E} \to \Omega^2_X \otimes \mathcal{E}$ vanishes

Claim (exercise): The conditions:

$\nabla_{f\theta} = f\nabla_\theta$, $\nabla_\theta(fs) = f\nabla_\theta(s) + \theta(f)\nabla_\theta(s)$, $[\nabla_\theta, \nabla_\eta] = \nabla_{[\theta,\eta]}$

imply that $\nabla : \Theta_X \times \mathcal{E} \to \mathcal{E}$ extends to left action $\mathcal{D}_X \times \mathcal{E} \to \mathcal{E}$!

Claim (exercise): $\mathcal{M}$ any left $\mathcal{D}_X$-mod $\longleftrightarrow$

$\mathcal{M}$ is qcoh. $\mathcal{O}_X$-mod + $\nabla : \mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$ s.t.:

1.) k-linear
2.) Leibniz
3.) $\nabla^2 = 0$