Characteristic varieties of $\mathcal{D}_X$-modules

<u>recall from last time:</u> X alg. variety$/k$, we have sheaf $\mathcal{D}_X$ which is $\mathcal{O}_X$-quasi-coherent with $\mathcal{O}_X$-coherent filtration $F_\bullet \mathcal{D}_X$

A left $\mathcal{D}_X$-module $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ together with left action $\mathcal{D}_X \times \mathcal{M} \to \mathcal{M}$.

$$\left(\Longleftrightarrow \quad \nabla : \mathcal{M} \to \Omega^1_X \otimes \mathcal{M}, \text{ flat connection operator} \right)$$

<u>examples:</u> – $\mathcal{M} = \mathcal{O}_X$

– $(\mathcal{E}, \nabla)$ vector bundle with connection $\rightsquigarrow \mathcal{E}$ left $\mathcal{D}_X$-mod.

– basic example for right $\mathcal{D}_X$-module: $\mathcal{M} := \omega_X := \Lambda^{\dim X} \Omega^1_{X/k}$

Consider Lie derivative on ω_X: For $U \subset X$, $v \in \Gamma(U, \Theta_X)$, $\omega \in \Gamma(U, \omega_X)$, and $v_1, \dots, v_n \in \Gamma(U, \Theta_X)$, we put

$$(\mathrm{Lie}_v \omega)(v_1, \dots, v_n) := v\left[\omega(v_1, \dots, v_n)\right] - \sum_{i=1}^{n} \omega(v_1, \dots, [v, v_i], \dots, v_n)$$

<u>Exercise:</u> 1) $\mathrm{Lie}_v(f\omega) = v(f) \cdot \omega + f \,\mathrm{Lie}_v(\omega) = \mathrm{Lie}_{(f\theta)}(\omega)$

2) $\mathrm{Lie}_{[v,\eta]}(\omega) = [\mathrm{Lie}_v, \mathrm{Lie}_\eta](\omega)$

notice: 2.) is similar to flatness of a connection

but 1.) shows that Lie is not a connection

(since we would need to have $\mathrm{Lie}_{fv}(\omega) = f\,\mathrm{Lie}_v(\omega)$)

Therefore, put: $\omega_X \times \Theta_X \longrightarrow \omega_X$

$$(\omega, v) \longmapsto -\mathrm{Lie}_v(\omega) = \omega \cdot v$$

and consider ω_X also as right $\mathcal{O}_X$-module $\Longrightarrow$

$$\omega \cdot \overbrace{v(f)}^{\in \mathcal{O}_X} \overset{1.)}{=\!=} \mathrm{Lie}_v(\omega \cdot f) - \mathrm{Lie}_v(\omega) \cdot f$$

$$= (\omega \cdot v) \cdot f - (\omega f) \cdot v$$

$$\Longleftrightarrow (\omega f) \cdot v = (\omega \cdot v) f - \omega \cdot v(f)$$

$$\omega \cdot [v, \eta] = -\mathrm{Lie}_{[v,\eta]}\, \omega = \mathrm{Lie}_\eta(\mathrm{Lie}_v\, \omega) - \mathrm{Lie}_v(\mathrm{Lie}_\eta\, \omega)$$

$$= \omega \cdot \eta \cdot v - \omega \cdot v \cdot \eta$$

This defines right $\mathcal{D}_X$-structure on ω_X.

Later: formulas for left / right transformation

Good filtrations / characteristic varieties:

very similarly to case of modules over $R/A_n(k)$:

$\mathcal{M}$ left $\mathcal{D}_X$-module, $F_j\mathcal{M}$ filtration by $\mathcal{O}_X$-coherent submodules s.t. $F_k\mathcal{D}_X \cdot F_j\mathcal{M} \subseteq F_{j+k}\mathcal{M}$, $\mathcal{M} = \bigcup_j F_j\mathcal{M}$

$F_\bullet\mathcal{M}$ is called good $\iff$ $gr^F_\bullet\mathcal{M}$ is locally finitely generated over $gr^F_\bullet\mathcal{D}_X$

<u>Rk</u>: already shown: $\forall U \subset X$, $D(A) := \Gamma(U, \mathcal{D}_X)$, then

$$M := \Gamma(U, \mathcal{M}) \text{ is } D(A)\text{-f.g.} \iff \exists \text{ good filtr. } F_\bullet M$$
$$\iff F_k D(A) \cdot F_j M \overset{!}{=} F_{j+k} M \quad \forall k,\ \forall j \gg 0$$

<u>Def</u>: $\mathcal{M}$ is called $\mathcal{D}_X$-coherent $\iff$ $\forall U \subset X$

$M = \Gamma(U, \mathcal{M})$ is $D(A) := \Gamma(U, \mathcal{D}_X)$ finitely generated

<u>Rk</u>: $\mathcal{M}$ $\mathcal{D}_X$-coherent $\not\Rightarrow$ $\mathcal{M}$ $\mathcal{O}_X$-coherent!

Let $F_\bullet \mathcal{M}$ be good filtration $\rightsquigarrow gr^F_\bullet \mathcal{M}$ is (locally f.g.) sheaf of $gr^F_\bullet \mathcal{D}_X \simeq \pi_* \mathcal{O}_{T^*X}$-algebras

Since π is affine, this corresponds to coherent $\mathcal{O}_{T^*X}$-module, denoted by $\widetilde{gr^F_\bullet \mathcal{M}}$.

<u>Def.</u>: Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, equipped with good filtration $F_\bullet \mathcal{M}$, then the characteristic variety $\mathrm{char}(\mathcal{M})$ is defined as $\mathrm{supp}(\widetilde{gr^F_\bullet \mathcal{M}})$, alternatively, $\mathrm{char}(\mathcal{M}) = V\left(\sqrt{\mathrm{ann}_{\mathcal{O}_{T^*X}} \widetilde{gr^F_\bullet \mathcal{M}}}\right)$.

<u>Rk</u>: as in local / A_n-case, one proves that $\mathrm{char}(\mathcal{M})$ does not depend on $F_\bullet \mathcal{M}$!

<u>ex</u>: $(\mathcal{E}, \nabla)$ VB with connection ($\rightsquigarrow \mathcal{E}$ is left $\mathcal{D}_X$-mod) in particular: $\mathcal{E}$ is $\mathcal{O}_X$-coherent. Put $F_j \mathcal{E} := \begin{cases} 0 & j < 0 \\ \mathcal{E} & j \geq 0 \end{cases}$

$\Rightarrow F_\bullet$ is good and $\mathrm{Ann}_{gr^F_\bullet \mathcal{D}_X} gr^F_\bullet \mathcal{E} = \bigoplus_{j \geq 1} gr^F_j \mathcal{D}_X$

$\Rightarrow \mathrm{char}(\mathcal{E}) = T^*_X X$ = zero section of π

Converse result:

Proposition: $\mathcal{M}$ $\mathcal{D}_X$-coherent. If $\mathcal{M}$ is $\mathcal{O}_X$-coherent, then $\mathcal{M}$ is $\mathcal{O}_X$-locally free (i.e. $\mathcal{M}$ is/corresponds to a vector bundle with flat connection)

Proof: By definition, $\mathcal{M}$ is $\mathcal{O}_X$-quasi-coherent.

Hence, it suffices to show: $\forall x \in X$: $\mathcal{M}_x$ is free $\mathcal{O}_{X,x}$-module of finite rank. Hence, show the following

Claim: Let $A\ (= \mathcal{O}_{X,x})$ regular local ring, $k \hookrightarrow A \twoheadrightarrow A/m_A$ and M left $D(A)$-module, which is f.g. over A

Then M is A-free of finite rk.

Proof of claim: Take $x_1, \dots, x_n \in m_A$ s.t. $m_A/m_A^2 = \bigoplus k[x_i]$ and consider coordinate vector fields $\partial_{x_1}, \dots, \partial_{x_n} \in \mathrm{Der}_k(A)$

for $f \in A \setminus \{0\}$, put $\mathrm{ord}(f) := \max\{e \mid f \in m_A^e\}$, $\mathrm{ord}(0) = +\infty$

Take generators $m_1, \dots, m_r$ of M s.t. $[m_1], \dots, [m_r]$ are basis of $M/m_A M$. Suppose $\exists$ relation $(f_1, \dots, f_r) \in A^r$ with $\sum_{i=1}^r f_i m_i = 0$, then $f_i \in m$ $\forall i$ (since $([m_i])_i$ lin. ind.)

Hence $\ell := \min\{\operatorname{ord}(f_1), \dots, \operatorname{ord}(f_r)\} \geq 1$. We have

$$0 = \partial_j\left(\sum_{i=1}^r f_i \cdot m_i\right) = \sum_{i=1}^r \partial_j(f_i) \cdot m_i + \sum_{i=1}^r f_i \cdot (\partial_j m_i)$$

$$\forall j, i: \quad \partial_j m_i = \sum_{k=1}^r \underbrace{a_{i,j,k}} \cdot m_k \implies \sum_{j=1}^r \left(\partial_i f_j + \sum_{k=1}^r a_{i,k,j} f_k\right) m_j = 0$$

Take j s.t. $\operatorname{ord}(f_j) = \ell$, then take i s.t. $\operatorname{ord}(\partial_i f_j) = \ell - 1$

$$\implies \operatorname{ord}\left[\left(\partial_i f_j + \sum_{k=1}^r a_{i,k,j} f_k\right)_j\right] = \ell - 1$$

Inductively, one gets relation of order $= 0$

$\lightning$ $[m_1], \dots, [m_r]$ lin. independent in $M/m_A M$ $\square$

Clearly, good filtrations exists locally on coherent $\mathcal{D}_X$-modules (by definition, and since $\exists F_\bullet\mathcal{M}$ good $\Leftrightarrow$ $\mathcal{M}$ locally f.g.). However, we also have:

<u>Lemma</u>: Let $\mathcal{M}$ be $\mathcal{D}_X$-coherent, then $\exists F_\bullet\mathcal{M}$ (global) good filtration by coherent $\mathcal{O}_X$-modules

<u>Proof</u>: enough to construct one single $\mathcal{O}_X$-coherent submodule $\mathcal{F} \subseteq \mathcal{M}$ that generates $\mathcal{M}$ as $\mathcal{D}_X$-module. Then we can put $F_k\mathcal{M} := F_k\mathcal{D} \cdot \mathcal{F}$ (check that each $F_k\mathcal{M}$ is $\mathcal{O}_X$-coherent).

X finite type$/K \Rightarrow X$ quasi-compact

$\exists$ finite affine cover $U_1, \dots, U_m \subset X$

$\forall i$: $\Gamma(U_i, \mathcal{M})$ is $\Gamma(U_i, \mathcal{D}_X)$-f.g., say by $m_1^{(i)}, \dots, m_{k_i}^{(i)}$

$$\Rightarrow \mathcal{F}_{U_i} := \sum_{j=1}^{k_i} \mathcal{O}_{U_i} \cdot m_j^{(i)} \text{ generates } \mathcal{M}_{|U_i} \text{ over } \mathcal{D}_{U_i}$$

<u>Fact:</u> X alg. variety $U \subset X$ open, $\mathcal{G}$ quasi-coherent $\mathcal{O}_X$-module. If $\exists$ coherent submodule $\mathcal{F}_U \subset \mathcal{G}_{|U}$

$\Rightarrow \exists$ coherent $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{G}$ s.t. $\mathcal{F}_{|U} = \mathcal{F}$

<u>in above situation:</u> $\exists$ $F_1, \ldots, F_m$ $\mathcal{O}_X$-coherent s.t.

$F_i{}_{|U_i} = \mathcal{F}_{U_i}$. Consider morphism

$$\varphi : F_1 \oplus \ldots \oplus F_m \longrightarrow \mathcal{M}$$
$$(e_1, \ldots, e_m) \longmapsto \sum e_i$$

and put $\mathcal{F} := \operatorname{im}(\varphi)$, then:

1.) $\mathcal{F}$ is $\mathcal{O}_X$-coherent

2.) $\mathcal{D}_X \cdot \mathcal{F} = \mathcal{M}$ $\square$

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next aim: discuss Lagrangian property of charact. variety

<u>reminder from symplectic geometry</u>:

1.) Symp. vector spaces: char$(K) \neq 2$, V K-vect. sp.

$\sigma : V \otimes_K V \to K$ bilinear, anti-symmetric + non-degenerate is called symplectic form

$\rightsquigarrow$ iso $V \to V^*$, $v \mapsto \sigma(-, v)$

$\rightsquigarrow \forall \vartheta \in V^* : \exists! H_\vartheta \in V : \vartheta(v) = \sigma(v, H_\vartheta) \ \forall v \in V$

$L \subset V \Rightarrow L^\perp := \{ v \in V \mid \sigma(v, w) = 0 \ \forall w \in L \}$

in part.: $w \in V \Rightarrow L_w := K \cdot w \subset V$

here we have $L_w \subset L_w^\perp$ since $\omega(w, w) = 0$

σ non-deg $\Rightarrow \dim L_w^\perp = \dim V - 1$

Claim: $L_w^\perp / L_w$ is again symplectic, but

$\dim L_w^\perp / L_w = \dim V - 2$, hence by induction:

$$\dim(V) = 2k \ !$$

<u>Def.</u>: (V, σ) symplectic, $W \subset V$ is called

(1) involutive iff $W^\perp \subset W$ (then $\dim W \geq \frac{1}{2} \dim V$)
(co-isotropic)

2.) isotropic iff $W^\perp \subset W$ (then $\dim W \leq \frac{1}{2}\dim V$)

3.) Lagrangian iff $W^\perp = W$ (then $\dim W = \frac{1}{2}\dim V$)

Rk: $W \subset V$ involutive, then W is Lagrangian iff $\dim W = \frac{1}{2}\dim V$

M non-sing. alg variety is called symplectic

$\exists\, \omega \in \Gamma(M, \Omega^2_{M/k})$ s.t. $\forall x \in M$ closed,

$\omega_x|_{T_x M}$ is symplectic form, $d\omega = 0$

Basic example: cotangent bundle: $\pi : \overset{M}{T^*X} \to X$

canonical projection.

1.) local description: $(x_1, \dots, x_n)$ local coordinates on $U \subset X$

$\Rightarrow \Omega^1_{X/k}|_U \simeq \bigoplus_{i=1}^n \mathcal{O}_U\, dx_i \rightsquigarrow (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ local

coord. on $\pi^{-1}(U)$, then $\omega = \sum_{i=1}^n d\xi_i \wedge dx_i$

2.) global description: $\exists$ non-vanishing section s

of $\pi^* \Omega^1_{X/k}$ (since $\mathrm{Tot}(\pi^*\Omega^1_{X/k}) = \Omega^1_{X/k} \times_X \Omega^1_{X/k}$)

Consider cotangent sequence for π:

$$0 \to \pi^* \Omega^1_{X/K} \xrightarrow{\varphi} \Omega^1_{T^*X/K} \to \Omega^1_{T^*X/X} \to 0$$

Put $\alpha := \varphi(s) \in \Gamma(T^*X, \Omega^1_{T^*X/K})$ Liouville form

ex.: locally $\alpha = \sum_{i=1}^{n} \xi_i dx_i$. Then $\omega := d\alpha$
$(\Rightarrow d\omega = 0)$

Hamiltonian vector fields: ω_x induces iso

$$\omega_x : T_x M \xrightarrow{\sim} T_x^* M \iff \sigma_x : \Theta_M \xrightarrow{\sim} \Omega^1_M$$

$$f \in \Gamma(M, \mathcal{O}_M) \Rightarrow H_f := \sigma_M^{-1}(df)$$

Poisson bracket: $f, g \in \Gamma(M, \mathcal{O}_M)$

$$\{f, g\} := H_f(g) = dg(H_f) = \sigma(H_f, H_g) \in \Gamma(M, \mathcal{O}_X)$$

ex: for $M = T^*X$, locally

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \xi_i} \cdot \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial \xi_i} \right)$$

Def.: (M,ω) symplectic, $Y\subset M$ reduced (not necess.-smooth) is called

$$\left.\begin{array}{c}\text{involutive}/\omega\text{-isotropic}\\ \text{isotropic}\\ \text{Lagrangian}\end{array}\right\}\ \forall x\in Y_{reg}: T_xY\subset T_xM \text{ is } \left\{\begin{array}{c}\text{involutive}/\omega\text{-isotropic}\\ \text{isotropic}\\ \text{Lagrangian}\end{array}\right.$$

example: $M:=T^*X$, take $Z\subset X$ reduced (possibly singular), consider $Z_{reg}\subset Z$.

Then $T^*_{Z_{reg}}X:=\{v\in T^*_xX \mid x\in Z_{reg},\ v_{|T_xZ_{reg}}=0\}$ is bundle over Z_{reg}, called conormal bundle.

exercise: $T^*_{Z_{reg}}X\subset T^*X$ is smooth Lagrangian
in part., $\dim T^*_{Z_{reg}}X=\dim X$

Prop: $T^*_ZX:=\overline{T^*_{Z_{reg}}X}$, then $T^*_ZX\subset T^*X$ is (possibly sing. Lagrangian)