<u>Recall from last time:</u> (M,ω) symplectic $\Leftrightarrow$ M smooth alg, $\omega \in \Gamma(M,\Omega^2_M)$

$d\omega = 0$, $\omega_{|T_xM}$ symplectic form $\forall x \in M$.

<u>example:</u> $M = T^*X$ cotangent bundle $\rightsquigarrow$ $\omega = d\alpha$, α locally

given as $\alpha = \sum_{i=1}^{n} \xi_i dx_i$, where $(x_1, \dots, x_n)$ local

coordinates on X.

Hamiltonian vector fields: ω_x induces iso

$$\omega_x : T_xM \xrightarrow{\cong} T_x^*M \iff \sigma_x : \Theta_M \xrightarrow{\cong} \Omega^1_M$$

$$f \in \Gamma(M, \mathcal{O}_M) \Rightarrow H_f := \sigma_M^{-1}(df)$$

<u>Poisson bracket:</u> $f, g \in \Gamma(M, \mathcal{O}_M)$

$$\{f,g\} := H_f(g) = dg(H_f) = \sigma(H_f, H_g) \in \Gamma(M, \mathcal{O}_X)$$

<u>ex:</u> for $M = T^*X$, locally

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial \xi_i} \cdot \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \cdot \frac{\partial g}{\partial \xi_i} \right)$$

<u>Def.</u>: (M,ω) symplectic, $Y \subset M$ reduced (not necess. smooth) is called

$\left.\begin{array}{l}\text{involutive}/\omega\text{-isotropic}\\ \text{isotropic}\\ \text{Lagrangian}\end{array}\right\}$ $\forall x \in Y_{reg}$: $T_xY \subset T_xM$ is $\left\{\begin{array}{l}\text{involutive}/\omega\text{-isotropic}\\ \text{isotropic}\\ \text{Lagrangian}\end{array}\right.$

<u>example</u>: $M := T^*X$, take $Z \subset X$ reduced (possibly singular), consider $Z_{reg} \subset Z$.

Then $T^*_{Z_{reg}}X := \{ v \in T^*_xX \mid x \in Z_{reg},\ v|_{T_xZ_{reg}} = 0 \}$ is bundle over Z_{reg}, called conormal bundle.

<u>exercise</u>: $T^*_{Z_{reg}}X \subset T^*X$ is smooth Lagrangian

in part., $\dim T^*_{Z_{reg}}X = \dim X$

Def: $T^*_ZX := \overline{T^*_{Z_{reg}}X}$, then $T^*_ZX \subset T^*X$ is (possibly singular) Lagrangian.

ex: $Z = X \Rightarrow \overline{T^*_X X} = T^*_X X = \{$zero section of $T^*X \to X\}$

$Z = \{x\},\ x \in X \Rightarrow \overline{T^*_Z X} = T^*_Z X = T^*_x X \subset T^*X$ Lag

$X = \mathbb{A}^2,\ Z = V(x^3 - y^2),\ T^*X \simeq \mathbb{A}^4_{x,y,z,w}$

$$T^*_Z X = V\left(x^3 - y^2,\ w^2x - \tfrac{4}{9}z^2,\ xz + \tfrac{3}{2}yw,\ wx^2 + \tfrac{2}{3}zy\right) \subset \mathbb{A}^4$$

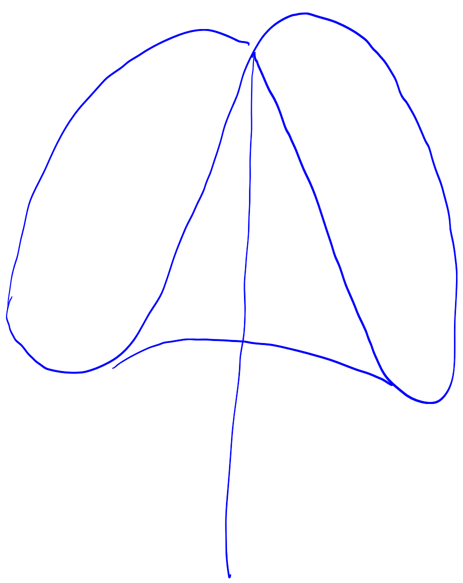
singular (at $0 \in \mathbb{A}^4$) Lagrangian

(not C.I.)

open Whitney umbrella

$$\begin{array}{ccc} T^*_Z X & & \mathbb{A}^4 \\ \downarrow & & \downarrow \\ V\left(w^2x - \tfrac{4}{9}z^2\right) & & \mathbb{A}^3_{x,z,w} \end{array}$$

ordinary Whitney umbrella



<u>Lemma</u>: (M,ω) symplectic alg. manifold, $Y \subset M$ reduced subspace, then

1.) Y is co-isotropic (at smooth pts.)

2.) $\{I_Y, I_Y\} \subset I_Y$

<u>exercise</u>

X smooth algebraic, $gr^F_\bullet \mathcal{D}_X \simeq \pi_* \mathcal{O}_{T^*X}$

describe Poisson bracket on $\pi_* \mathcal{O}_{T^*X}$ in terms of $\mathcal{D}_X$ (rather $gr^F_\bullet \mathcal{D}_X$)

For $P \in F_k \mathcal{D}_X$, write $\sigma_k(P) \in gr^F_k \mathcal{D}_X$ for its class in $F_k \mathcal{D}_X / F_{k-1} \mathcal{D}_X$ (principal symbol)

If $P \in F_k \mathcal{D}_X$, $Q \in F_\ell \mathcal{D}_X \underset{\text{seen}}{\overset{\text{already}}{\Longrightarrow}} [P,Q] \subset F_{k+\ell-1} \mathcal{D}_X$

<u>exercise</u>: $\sigma_{k+\ell-1}([P,Q]) = \{\sigma_k(P), \sigma_\ell(Q)\}$

Let $\mathcal{M} \in \mathrm{Coh}(\mathcal{D}_X)$ (coherent $\mathcal{D}_X$-mod.)

$F_\bullet \mathcal{M}$ good filtration.

<u>Lemma</u>: The ideal $\mathrm{Ann}_{\mathrm{gr}^F \mathcal{D}_X}(\mathrm{gr}^F \mathcal{M})$ is involutive (closed under $\{ \, , \}$).

<u>Proof</u>: Statement is local, so assume

$A = \Gamma(U, \mathcal{O}_X)$, $R = D(A) = \Gamma(U, \mathcal{D}_X)$, M f.g. left R-mod

$F_\bullet M$, $\mathrm{gr}^F_\bullet M$ f.g. over $S = \mathrm{gr}^F_\bullet R$.

<u>Show</u>: $I := \mathrm{ann}_S \, \mathrm{gr}^F M$ satisfies $\{I, I\} \subset I$

<u>Take</u>: $P \in F_i R$, $Q \in F_j R$ s.t. $\sigma_i(P), \sigma_j(Q) \in I$

$\Rightarrow \forall k$: $P \cdot F_k M \subset F_{k+i-1} M$, $Q \cdot F_k M \subseteq F_{k+j-1} M$

$\Rightarrow [P, Q] \cdot F_k M \subseteq P F_{k+j-1} M + Q \cdot F_{k+i-1} M \subset F_{k+i+j-2} M$

$= F_{(i+j-1)+k-1} M \Rightarrow \sigma_{i+j-1}([P, Q]) = \{\sigma_i(P), \sigma_j(Q)\} \overset{!}{\in} I$ $\square$

Theorem (Gabber): M $\mathcal{D}_X$-coherent, then any prime minimal over $\sqrt{\mathrm{ann}_{gr_\bullet^F \mathcal{D}_X}\, gr_\bullet^F M}$ is involutive

Corollary: Any component $Y \subset \mathrm{char}(M)$ is a co-isotropic subspace of T^*X.

Rk: – Neither the theorem, nor the corollary follows from above lemma

– Gabber's theorem is purely algebraic, and most general: $gr_\bullet^F \mathcal{D}_X$ must be commutative, noetherian, but not necessarily regular

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Holonomic $\mathcal{D}_X$-modules: Gabber's theorem $\Longrightarrow$

$\dim \mathrm{char}(M) \geq n := \dim X$

Def: $M \in \mathrm{Coh}(\mathcal{D}_X)$ is called holonomic if

1.) $M = 0$ or

2.) $M \neq 0$ and $\dim(\mathrm{char}(M)) = n$

By Gabber's theorem, if $M \neq 0$ is holonomic, then each component is Lagrangian in T^*X.

Moreover, since $\sqrt{\operatorname{ann} \operatorname{gr}^F M}$ is homogeneous, $\operatorname{char}(M)$ is conical, i.e. $\mathbb{C}^*$-equivariant ($\mathbb{C}^*$ acting by scaling fibres of $\pi: T^*X \to X$)

Geometry of (components of) $\operatorname{char}(M)$:

<u>Proposition 12.4.:</u> Let $W \subset T^*X$ be irreducible, conical and Lagrangian.

Then $Y := \pi(W) \subset X$ is an irreducible subvariety and $W = T_Y^*X \left(:= \overline{T^*_{Y_{reg}} X} \right)$

<u>Proof:</u> local statement $\rightsquigarrow$ $X = \operatorname{Spec}(A)$ affine

$T^*X \simeq X \times \mathbb{A}^n_k$. $W \subset X \times \mathbb{A}^n_k$ conical

$\rightsquigarrow$ $I := I(W) \subset A[\xi_1, \dots, \xi_n]$ ξ-homog.

Consider $\hat{W} := V(I) \subset \mathbb{P}^n_A$, and projection $\hat{\pi} : \mathbb{P}^n_A \to X$, then $Y = \pi(W) = \hat{\pi}(\hat{W})$, but since $\hat{\pi}$ is projective $\Rightarrow Y \subset X$ is closed and irreducible. To show $W = T^*_Y X$. Both sides are irreducible of dim $= n$, hence

to show: (generic point of W) $\in T^*_{Y_{reg}} X$.

Take $(\bar{x}, \bar{\xi}) \in W$ general non-sing, then $\bar{x} \in Y_{reg}$ and $\pi_{|W}$ smooth at $(\bar{x}, \bar{\xi})$, i.e.

$T\pi_{(\bar{x},\bar{\xi})} : T_{(\bar{x},\bar{\xi})} W \to T_{\bar{x}} Y$ surjective.

Locally around $\bar{x} \in X$ with coord. $x_1, \dots, x_n$,

let $Y = V(x_{k+1}, \dots, x_n) \Longrightarrow T^*_{Y_{reg}} X = V(\xi_1, \dots, \xi_k, x_{k+1}, \dots, x_n)$

we have to show: $\hat{\xi}_1, \dots, \hat{\xi}_k = 0$

(so that $(\hat{x}, \hat{\xi}) \in T^*_{Y_{reg}} X$)

<u>Claim</u>: $\alpha_x|_{W_{reg}} = 0$, <u>Pf</u>: $\forall v \in T(W_{reg})$

$\alpha_x(v) = \omega(\iota(\alpha_x), v)$, where $\iota: T^*(T^*X) \xrightarrow{\simeq} T(T^*X)$ is iso given by ω

locally $\alpha_x = \sum \xi_i dx_i \Rightarrow \iota(\alpha_i) = \sum \xi_i \partial_{x_i}$

W conical $\Rightarrow \iota(\alpha_i)$ tangent to $W \Rightarrow \omega(\iota(\alpha_x), v) = 0$

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choose section $s: Y_{reg} \hookrightarrow W_{reg}$ of $W_{reg} \to Y_{reg}$

and compose with $i: W_{reg} \hookrightarrow W \hookrightarrow T^*X$

hence $(i \circ s)^* \alpha_x \underset{\substack{\text{since } x_i = 0 \\ \text{on } Y_{reg} \text{ for } i \in \{k+1, \dots, n\}}}{=} \sum_{i=1}^{k} \xi_i(x') dx_i = 0$ with

$x' = (x_1, \dots, x_k) \in Y_{reg}$

$\Rightarrow \forall x' \in Y_{reg}$, $\xi_1(x') = \dots = \xi_k(x') = 0$

hence for $x' = \tilde{x}$, we get $\tilde{\xi}_1 = \dots = \tilde{\xi}_k = 0$ $\square$

<u>Consequence</u>: $0 \neq M$ holonomic $\Rightarrow \operatorname{char}(M) = \bigcup_{i=1}^{m} T^*_{Y_i} X$

Case 1: $\exists i : Y_i = X$, put $U = X \setminus \bigcup_{j \neq i} Y_j$

$\Rightarrow \mathcal{M}_{|U}$ is VB + connection (since $\mathrm{char}(\mathcal{M}_{|U}) = T^*_U X$ 0-section)

Case 2: $\forall i : Y_i \subsetneq X \Rightarrow \mathcal{M}_{|U} = 0$

i.e. $\mathrm{supp}(\mathcal{M}) \subset \bigcup Y_i \subsetneq X$

in both cases: A holonomic $\mathcal{D}_X$-module is generically a vector bundle + flat connection

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Rk on duality: $\mathcal{M}$ $\mathcal{D}_X$-coherent (left or right)

$\Rightarrow \forall k$: $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ is (right or left) holonomic

on $U \subset X$ affine: $\Gamma(U, \mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)) = \mathrm{Ext}^k_{D(A)}(M, D(A))$

with $A = \Gamma(U, \mathcal{O}_X) \rightsquigarrow \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0 \ \forall i > n$

and $j(\mathcal{M}) + \dim \mathrm{char}(\mathcal{M}) = 2n$, where

$$j(M) = \min\{ j \mid \operatorname{Ext}^j_{\mathcal{D}_X}(M, \mathcal{D}_X) \neq 0 \}$$

Def: $0 \neq M$ holonomic $\left(\Rightarrow \operatorname{Ext}^j_{\mathcal{D}_X}(M, \mathcal{D}_X) = 0 \ \forall j \neq n \right)$

define $M^* = \operatorname{Ext}^n_{\mathcal{D}_X}(M, \mathcal{D}_X)$ to be its holonomic dual module

(later: left/right conversion)