Direct and inverse images of $\mathcal{D}$-modules

setup: $f: X \to Y$ map between smooth alg. var.

$\mathcal{M}$ $\mathcal{D}_X$-module, $\mathcal{N}$ $\mathcal{D}_Y$-module, define

$f_+\mathcal{M}$ (direct image), $f^+\mathcal{N}$ (inverse image)

If f is smooth, $f^+ = f^*$; $f_+ \rightsquigarrow$ GM-connection

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motivating examples: Let $X = \mathbb{A}^{n-1}$, $Y = \mathbb{A}^n$, $i: X \hookrightarrow Y$.

Let M be $D_{\mathbb{A}^{n-1}}$-module, and consider naive sheaf-theoretic pushforward $f_* M$. This is <u>not</u> a $D_{\mathbb{A}^n}$-module, since when considering M as $D(K[x_1, \dots, x_{n-1}])$ module, then $f_* M$ corresponds to $K[x_1, \dots, x_n]$-module where x_n acts trivially. Then $\partial_n \cdot M = 0$ is not allowed, since $[\partial_n, x_n] = 1 \neq 0$!

How to define direct images of D-modules. Look at distributions (which is a right D-module):

Let $\mathcal{F}_k$ some space of test functions on $\mathbb{R}^k$ and

$D \in \mathcal{F}_k'$, consider $i: \mathbb{R}^k \hookrightarrow \mathbb{R}^n$, $(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$,

then $i_* D$ is defined by $\langle i_* D, \varphi \rangle := \langle D, \varphi_{|\mathbb{R}^k} \rangle \quad \forall \varphi \in \mathcal{F}_n$

Put $\mathrm{Ann}_{A_k}(D) := \{ P \in A_k \mid D.P = 0 \}$ (right A_k-ideal)

(recall right action on $\mathcal{F}_k'$: $\langle D.P, \varphi \rangle := \langle D, P\varphi \rangle$)

and right module $A_k / \mathrm{Ann}_{A_k}(D)$)

Question: What is $\mathrm{Ann}_{A_n}(i_* D) := \{ Q \in A_n \mid i_* D . Q = 0 \}$?

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ok: since $x_{k+i} \cdot \varphi = 0 \quad \forall \varphi \in \mathcal{F}_k$, we have

for any $Q = x_{k+1} Q_{k+1} + \dots + x_n Q_n \in (x_{k+1}, \dots, x_n) A_n$ that

$$\langle i_* D . Q, \varphi \rangle = \sum_{i=k+1}^{n} \langle i_* D, x_i Q_i \varphi \rangle = \sum_i \langle D, \underbrace{(x_i Q_i \varphi)_{|\mathbb{R}^k}}_{=0} \rangle = 0$$

$\Rightarrow (x_{k+1}, \dots, x_n) A_n \subset \mathrm{Ann}_{A_n}(i_* D)$

Moreover: $\forall Q \in A_n$, write $Q = \sum_{i=k+1}^{n} x_i Q_i + \sum_{\alpha \in \mathbb{N}^{n-k}} P^\alpha \cdot \partial_{x_{k+1}}^{\alpha_{k+1}} \cdots \partial_{x_n}^{\alpha_n}$

where $P^\alpha \in A_k$. Now let $\varphi \in F_k$, $\eta \in F_{n-k}$, then

$$\langle i_p D \cdot Q, \varphi \cdot \eta \rangle = \sum_{\alpha \in \mathbb{N}^{n-k}} \langle i_p D \cdot P^\alpha \partial_{k+1}^{\alpha_{k+1}} \cdots \partial_n^{\alpha_n}, \varphi \cdot \eta \rangle$$

$$= \sum_\alpha \langle i_p D, (P^\alpha \partial_{k+1}^{\alpha_{k+1}} \cdots \partial_n^{\alpha_n})(\varphi \cdot \eta) \rangle$$

$$= \sum_\alpha \langle i_p D, P^\alpha(\varphi) \cdot (\partial_{k+1}^{\alpha_{k+1}} \cdots \partial_n^{\alpha_n})(\eta) \rangle$$

$$= \sum_\alpha \langle D, \underbrace{P^\alpha(\varphi)|_{\mathbb{R}^k}}_{\text{since } \varphi \in \mathbb{R}^k} \cdot \underbrace{(\partial_{k+1}^{\alpha_{k+1}} \cdots \partial_n^{\alpha_n})(\eta)|_{\mathbb{R}^k}}_{= (\partial_{k+1}^{\alpha_{k+1}} \cdots \partial_n^{\alpha_n} \eta)(0) \text{ since } \eta \in \mathbb{R}^{n-k}} \rangle$$

$$= \sum_{\alpha \in \mathbb{N}^{n-k}} (\partial_{k+1}^{\alpha_{k+1}} \cdots \partial_n^{\alpha_n} \eta)(0) \cdot \langle D, P^\alpha(\varphi) \rangle = 0$$

if $Q \in \mathrm{Ann}_{A_n}(i_p D)$

chose φ such that only one summand is $\neq 0$

$$\Rightarrow \quad 0 = \langle D, P^\alpha(\varphi)\rangle = \langle D.P^\alpha, \varphi\rangle$$

$\forall \alpha \in \mathbb{N}^{n-k}$, $\forall \varphi \in \mathcal{F}_k$. Hence: $\forall \alpha \in \mathbb{N}^{n-k}$: $P^\alpha \in \mathrm{Ann}_{A_k}(D)$

clearly: $P \in \mathrm{Ann}_{A_k}(D) \Rightarrow P \cdot A_n \subset \mathrm{Ann}_{A_n}(i_* D)$

<u>Conclusion</u>: If D is solution to system given by $A_k / (P_1, \dots, P_r) \cdot A_k$, then $i_* D$ is solution to system $A_n / (P_1, \dots, P_r, x_{k+1}, \dots, x_n) \cdot A_n$.

<u>Moral</u>: Direct image can be "easily" defined for right modules

Before giving general definitions, we need to discuss left / right transformations.

<u>Recall</u>: X smooth alg., ω_X is right $\mathcal{D}_X$-module by

$$\omega . v := - \mathrm{Lie}_v \, \omega \qquad \forall v \in \Theta_X$$

<u>Proposition</u>: (a) $\mathcal{M}, \mathcal{N}$ left $\mathcal{D}_X$-modules $\Longrightarrow$

$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is left module via

$$v.(m \otimes n) := vm \otimes n + m \otimes vn$$

(b) $\mathcal{M}'$ right $\mathcal{D}_X$-mod., $\mathcal{N}$ left $\mathcal{D}_X$-mod

$\Longrightarrow \mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N}$ is right $\mathcal{D}_X$-mod via

$$(m \otimes n).v := m.v \otimes n + m \otimes v.n$$

$$\Big(\text{notice } (m \otimes n).(fv) = m.(fv) \otimes n + m \otimes (fv).n$$

$$= (m \cdot f).v \otimes n + m \cdot f \otimes vn$$

$$= (m \cdot f \otimes n).v = \big((m \otimes n).f\big).v\Big)$$

does not work for tensor product of 2 right modules

(c) $\mathcal{M}, \mathcal{N}$ left $\mathcal{D}_X$-modules $\Longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is left $\mathcal{D}_X$-mod via

$$v.\phi := \big(m \longmapsto v.(\phi(m)) - \phi(v.m)\big)$$

(d) $\mathcal{M}', \mathcal{N}'$ right $\mathcal{D}_X$-mod $\Rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ is left $\mathcal{D}_X$-mod via

$$v.\phi := \left(m \mapsto \phi(m.v) - \phi(m).v \right)$$

(e) $\mathcal{M}$ left, $\mathcal{N}'$ right $\mathcal{D}_X$-module

$\Rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}')$ is right $\mathcal{D}_X$-module via

$$\phi.v := \left(m \mapsto \varphi(m).v - \varphi(v.m) \right)$$

(e.g. check that $\phi.(fv) = (f\cdot\phi).v$)

<u>Corollary</u>: Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module, then $\mathcal{M}^r := \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a right $\mathcal{D}_X$-module (because of (b)). If $\mathcal{N}'$ is a right $\mathcal{D}_X$-module, then $\mathcal{N}'^{,\ell} := \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N}')$ is a left $\mathcal{D}_X$-mod. We have $(\mathcal{M}^r)^\ell \simeq \mathcal{M}$ and $(\mathcal{N}'^{,\ell})^r \simeq \mathcal{N}'$.

Definition of direct image for closed embeddings:

Transfer module: let $Y \xrightarrow{i} X$ (smooth, alg. varieties)

Define $\mathcal{D}_{Y \to X} := \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X$, this is a

$(\mathcal{D}_Y, i^{-1}\mathcal{D}_X)$-bimodule:

1.) right $i^{-1}\mathcal{D}_X$: $v \in i^{-1}\mathcal{D}_X$, $f \in \mathcal{O}_Y$, $\varphi \in i^{-1}\mathcal{D}_X$: $(f \otimes \varphi).v = f \otimes \varphi \cdot v$

2.) left $\mathcal{D}_Y$: for any $g \in i^{-1}\mathcal{O}_X$, any $f \in \mathcal{O}_Y$, $\varphi \in i^{-1}\mathcal{D}_X$, define action: $(f \otimes \varphi)(g) := f \cdot (\varphi(g) \circ i) \in \mathcal{O}_Y$, hence,

$\mathcal{D}_{Y \to X}$ is sheaf of differential operators from $i^{-1}\mathcal{O}_X$ to $\mathcal{O}_Y$

then put $\forall\, \delta \in \mathcal{D}_Y$, $f \in \mathcal{O}_Y$, $\varphi \in i^{-1}\mathcal{D}_X$:

$$\delta . (f \otimes \varphi) := \Big[g \longmapsto \delta\big((f \otimes \varphi)(g)\big) \Big]$$

Local expression when $i : Y \hookrightarrow X$ is closed embedding

$Y = \{x_{r+1} = \dots = x_n = 0\}$, consider $y_1 := x_1, \dots, y_r := x_r$ as local coordinates on Y

<u>Lemma</u>: $\exists$ injective homomorphism of left $\mathcal{D}_Y$-modules $\mathcal{D}_Y \hookrightarrow \mathcal{D}_{Y\to X}$ and $\mathcal{D}_{Y\to X}$ is $\mathcal{D}_Y$-locally free (of $\mathrm{rk} = \infty$)

<u>Proof</u>: Consider section $1\otimes 1$ of $\mathcal{D}_{Y\to X} = \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X$.

Then $P = \sum\limits_{\alpha\in\mathbb{N}^r} f_\alpha \partial_y^\alpha \in \mathcal{D}_Y \mapsto P\cdot(1\otimes 1) = \sum\limits_\alpha f_\alpha \otimes \partial_{x_1}^{\alpha_1}\cdots\partial_{x_r}^{\alpha_r}$

is injective module homomorphism.

to show: $\mathcal{D}_{Y\to X}$ is $\mathcal{D}_Y$-locally free.

chose local coordinates as above, let arbitrary element $Q\in\mathcal{D}_X$ be given as $Q = \sum\limits_{\beta\in\mathbb{N}^n} g_\beta \partial^\beta$, write $\overline{g}_\beta := g_\beta|_Y \in \mathcal{O}_Y$. Then $\forall f\in\mathcal{O}_Y$:

$$f\otimes Q = \sum_{\beta\in\mathbb{N}^n} f\otimes g_\beta\partial^\beta = \sum_{\beta_{r+1},\dots,\beta_n} \underbrace{\left(\sum_{\beta_1\dots\beta_r} \overline{g}_\beta \otimes \partial_{y_1}^{\beta_1}\cdots\partial_{y_r}^{\beta_r}\right)}_{\in\mathcal{D}_Y} \cdot \partial_{x_{r+1}}^{\beta_{r+1}}\cdots\partial_{x_n}^{\beta_n}$$

Hence we have locally: $\mathcal{D}_{Y\to X} = \mathcal{D}_Y[\partial_{x_{r+1}}, \ldots, \partial_{x_n}]$

$$\cong \mathcal{D}_Y \otimes_K K[\partial_{x_{r+1}}, \ldots, \partial_{x_n}]$$

<u>Def</u>: $i: Y \hookrightarrow X$ closed embedding, $\mathcal{M}$ a <u>right</u> $\mathcal{D}_Y$-module, then the direct image is defined as $i_+\mathcal{M} := i_*(\mathcal{M} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$

<u>Notice</u>: $\mathcal{M}$ right $\mathcal{D}_Y$-mod, $\mathcal{D}_{Y\to X}$ left $\mathcal{D}_Y$-mod

$\rightsquigarrow \mathcal{M} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ is well-defined. It carries right $i^{-1}\mathcal{D}_X$-structure due to right $i^{-1}\mathcal{D}_X$-structure of $\mathcal{D}_{Y\to X}$ $\rightsquigarrow i_*(\mathcal{M} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$ is right $\mathcal{D}_X$-mod.

<u>moreover</u>: locally, since $\mathcal{D}_{Y\to X} = \mathcal{D}_Y[\partial_{x_{r+1}}, \ldots, \partial_{x_n}]$, we have $i_+\mathcal{M} \cong i_*\mathcal{M}[\partial_{x_{r+1}}, \ldots, \partial_{x_n}]$

<u>finally</u>: i_+ is exact since both i_* and $- \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X}$ are exact (recall that $\mathcal{D}_{Y\to X}$ is $\mathcal{D}_Y$-locally free)

<u>example</u>: $M \simeq \mathcal{D}_Y$, then $i_+\mathcal{D}_Y = i_*(\mathcal{D}_Y \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$

$\simeq i_*\mathcal{D}_{Y\to X} \simeq i_*\left(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X\right)$

Consider morphism of right $\mathcal{D}_X$-modules

$$\phi : \mathcal{D}_X \longrightarrow i_+\mathcal{D}_Y \simeq i_*\left(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X\right)$$
$$P \longmapsto 1 \otimes P$$

ϕ surjective, and $\ker(\phi) = \mathcal{I}_Y \cdot \mathcal{D}_X$, hence:

$$i_+\mathcal{D}_Y \simeq \mathcal{D}_X / \mathcal{I}_Y \cdot \mathcal{D}_X$$

This holds more generally for cyclic modules,

<u>ex</u>: distributions: $Y = \mathbb{A}^r_k$, $X = \mathbb{A}^n_k$, $i : \mathbb{A}^r_k \to \mathbb{A}^n_k$

$M = A_r / (P_1, \dots, P_m) \cdot A_r$ right module satisfied by distribution $D \in F_r'$

we have exact sequence $A_r^m \xrightarrow{(P_1, \dots, P_m)\cdot} A_r \to M \to 0$

and we know from previous considerations:

$$i_+ A_r \simeq A_n / (x_{r+1}, \ldots, x_n) \cdot A_n$$

Hence (using exactness of i_+):

$$i_+ M = i_+ \operatorname{coker}\left((P_1, \ldots, P_m) \cdot - : A_r^m \to A_r \right)$$

$$= \operatorname{coker}\left((P_1, \ldots, P_m) \cdot - : i_+ A_r^m \longrightarrow i_+ A_r \right)$$

$$= \operatorname{coker}\left[\left(A_n / (x_{r+1}, \ldots, x_n) A_n \right)^m \longrightarrow A_n / (x_{r+1}, \ldots, x_n) A_n \right]$$

$$\simeq A_n / (P_1, \ldots, P_m, x_{r+1}, \ldots, x_n) \cdot A_n$$

This agrees with previous computations.