Kashiwara's equivalence

recall from last time: $i: Y \hookrightarrow X$ closed embedding,

$\mathcal{M}$ right $\mathcal{D}_Y$-module, define $i_+\mathcal{M} := i_*(\mathcal{M} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y\to X})$

we have injection $i_*\mathcal{M} \hookrightarrow i_+\mathcal{M}$, locally $i_+\mathcal{M} = i_*\mathcal{M}[\partial_{x_{r+1}}, \dots, \partial_{x_n}]$

aim: study $\mathrm{char}(i_+\mathcal{M})$.

Lemma: $\mathcal{M}$ is coherent right $\mathcal{D}_Y$-module $\Rightarrow$ $i_+\mathcal{M}$ is coherent right $\mathcal{D}_X$-module

Proof: $\exists$: $\mathcal{F} \in \mathrm{Coh}(\mathcal{O}_Y)$: $\mathcal{F} \subset \mathcal{M}$, $\mathcal{M} = \mathcal{F} \cdot \mathcal{D}_Y$

since locally $i_+\mathcal{M} = i_*\mathcal{M}[\partial_{x_{r+1}}, \dots, \partial_{x_n}]$, we have

$i_+\mathcal{M} = i_*\mathcal{F} \cdot \mathcal{D}_X \Rightarrow i_+\mathcal{M}$ is coherent right $\mathcal{D}_X$-coherent

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Question: $\mathrm{char}(i_+\mathcal{M}) = ?$, need good filtration on $i_+\mathcal{M}$

let good filtration $F_\bullet\mathcal{M}$ be given, then

$i_* F_k \mathcal{M} \subset i_*\mathcal{M} \subset i_+\mathcal{M} \cong i_*\mathcal{M}[\partial_{x_{k+1}}, \dots, \partial_{x_n}]$

Define $F_k i_+ \mathcal{M} := i_* F_k \mathcal{M} + i_* F_{k-1} \mathcal{M} \cdot F_1 \mathcal{D}_X + i_* F_{k-2} \cdot F_2 \mathcal{D}_X + \ldots$

clear: $F_j \mathcal{M} = 0 \ \forall j < 0 \Rightarrow$ finite sum $\Rightarrow$ $F_k i_+ \mathcal{M}$ is $\mathcal{O}_X$-coherent

to show: $F_\bullet i_+ \mathcal{M}$ is good: work locally as before

$U \subset X$ affine, $A = \Gamma(U, \mathcal{O}_X)$, $Y \cap U = V(I)$; $I = (x_{r+1}, \ldots, x_n)$

$M = \Gamma(U, \mathcal{M})$, $i_+ M \cong M \otimes_K K[\partial_{r+1}, \ldots, \partial_n]$

$$F_k i_+ M = \sum_{\alpha \in \mathbb{N}^{n-r}} F_{k-|\alpha|} M \otimes \partial_{r+1}^{\alpha_{r+1}} \cdot \ldots \cdot \partial_n^{\alpha_n}$$

$$\Rightarrow gr_k^F(i_+ M) = \bigoplus_{\alpha \in \mathbb{N}^{n-r}} gr^F_{k-|\alpha|} \otimes \partial_{r+1}^{\alpha_{r+1}} \cdot \ldots \cdot \partial_n^{\alpha_n}$$

$$\leadsto gr_\bullet^F(i_+ M) = gr_\bullet^F M \otimes_K K[\partial_{r+1}^{\alpha_{r+1}}, \ldots, \partial_n^{\alpha_n}]$$

this is $gr^F D(A) \simeq A[\partial_1, \ldots, \partial_n]$-module.

We know that $gr_\bullet^F M$ is $(A/I)[\partial_1, \dots, \partial_r]$-f.g.,

hence $gr_\bullet^F(i_+ M) \simeq gr_\bullet^F M \underset{\frac{A}{I}[\partial_1 \dots \partial_r]}{\otimes} A/I[\partial_1, \dots, \partial_n]$

Iso of $A[\partial_1, \dots, \partial_n]$-modules ($I$ acts trivially)

$\Rightarrow gr_\bullet^F(i_+ M)$ is $A[\partial_1, \dots \partial_n]$-f.g.

$\Rightarrow F_\bullet(i_+ M)$ is good

<u>geometric interpretation</u>: For embedding $Y \overset{i}{\hookrightarrow} X$

we have conormal morphism $i^*\Omega_{X/K} \xrightarrow{di} \Omega_{Y/K}$.

Written as total spaces of bundles:

$$\begin{tikzcd} Y \times_X T^*X \arrow[r, "di"] \arrow[d, "p_2"] & T^*Y \\ T^*X & \end{tikzcd} \qquad (*)$$

notice: - di is smooth of rel. dim $\dim X - \dim Y$

- p_2 is closed embedding (since $T^*X = X \times_X T^*X$)

consider $gr^F M$ (coherent $\pi_* \mathcal{O}_{T^*Y}$-module) and associated $\mathcal{O}_{T^*Y}$-module $\widetilde{gr^F_\bullet \mathcal{M}}$, and similarly $\widetilde{gr^F_\bullet (i_+\mathcal{M})}$ (coherent $\mathcal{O}_{T^*X}$-module) then we obtain:

$$\widetilde{gr^F_\bullet (i_+\mathcal{M})} = \left(p_{2,*}(di)^*\right)\left(\widetilde{gr^F_\bullet \mathcal{M}}\right)$$

$\boxed{\text{Proof}}$: exercise: show this locally over $U \subset X$ affine as above, where $(*)$ corresponds to

$$\begin{tikzcd}
(A/I)[\partial_1,\dots,\partial_n] & (A/I)[\partial_1,\dots,\partial_r] \arrow[l] \\
A[\partial_1,\dots,\partial_n] \arrow[u] &
\end{tikzcd}$$

$\square$

Corollary: in above situation, we have

(a) $\quad \operatorname{char}(i_+\mathcal{M}) = p_2\left((di)^{-1}(\operatorname{char}(\mathcal{M}))\right)$

(b) $\quad \dim \operatorname{char}(i_+\mathcal{M}) = \dim \operatorname{char}(\mathcal{M}) + \dim X - \dim Y$

Proof: (a) clear since $\operatorname{char}(i_+\mathcal{M}) = \operatorname{supp}(\widetilde{\operatorname{gr}^F i_+\mathcal{M}})$
and $\operatorname{char}(\mathcal{M}) = \operatorname{supp}(\widetilde{\operatorname{gr}^F \mathcal{M}})$

(b) follows from (a) since di is smooth and p_2 is closed embedding

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Consequences: – $\mathcal{M}$ holonomic $\Rightarrow \dim \operatorname{char}(\mathcal{M}) = \dim Y$
$\Rightarrow \dim(i_+\mathcal{M}) = \dim X \Rightarrow i_+\mathcal{M}$ holonomic

– new proof of $\dim \operatorname{char}(\mathcal{M}) \geq \dim X$ without using symplectic geometry: locally $Y \overset{i}{\subset} X = \mathbb{A}^n$

$\Rightarrow \dim \operatorname{char}(\mathcal{M}) - \dim Y = \dim \operatorname{char}(i_+\mathcal{M}) - n$

but $i_+\mathcal{M}$ corresponds to module over A_n

$\xRightarrow{\text{Bernstein's inequality}} \dim \operatorname{char}(i_+\mathcal{M}) \geq n$

hence $\dim \operatorname{char}(\mathcal{M}) \geq \dim Y$

Lemma: i_+ is functor

$$i_+ \text{ (coherent right } \mathcal{D}_Y\text{-mod's)} \longrightarrow \text{(coherent right } \mathcal{D}_X\text{-mod's)}$$

Def: M coherent $\mathcal{D}_X$-mod, put

$$\mathrm{supp}(M) := \pi\big(\underbrace{\mathrm{char}(M)}_{\subset T^*X}\big) \subset X$$

closed, since $\mathrm{char}(M) \subset T^*X$ conical

Theorem (Kashiwara's equivalence):

i_+ induces equivalence of cat's:

$$i_+ : \text{(coherent right } \mathcal{D}_Y\text{-mod's)} \longrightarrow \text{(coherent right } \mathcal{D}_X\text{-mod's with support} \subset Y)$$

Proof: define inverse functor: let N be right $\mathcal{D}_X$-mod with $\mathrm{supp}(M) \subset Y$ and consider

$$i^{\#}\mathcal{N} := \mathcal{H}om_{i^{-1}\mathcal{D}_X}(\mathcal{D}_{Y\to X}, i^{-1}\mathcal{N})$$

<u>notice</u>:

- both $\mathcal{D}_{Y\to X}$, $i^{-1}\mathcal{N}$ are right $i^{-1}\mathcal{D}_X$-modules, so one can take $\mathcal{H}om$
- since $\mathcal{D}_{Y\to X}$ has left $\mathcal{D}_Y$-structure, we obtain right $\mathcal{D}_Y$-structure on $i^{\#}\mathcal{N}$ by putting: $\forall\, v \in \Theta_Y$:

$$\phi . v := \big(\delta \mapsto \phi(v.\delta)\big)$$

recall that $\mathcal{D}_{Y\to X} = \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X$, we obtain

$$i^{\#}\mathcal{N} := \mathcal{H}om_{i^{-1}\mathcal{D}_X}\Big(\mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X, i^{-1}\mathcal{N}\Big) =$$

$$\mathcal{H}om_{i^{-1}\mathcal{O}_X}\Big(\mathcal{O}_Y, \underbrace{\mathcal{H}om_{i^{-1}\mathcal{D}_X}(i^{-1}\mathcal{D}_X, i^{-1}\mathcal{N})}_{i^{-1}\mathcal{N}}\Big)$$

$$= \mathcal{H}om_{i^{-1}\mathcal{O}_X}(\mathcal{O}_Y, i^{-1}\mathcal{N})$$

Apply $\mathcal{H}om_{i^{-1}\mathcal{O}_X}(-, i^{-1}\mathcal{N})$ to

$$0 \to i^{-1}\mathcal{I}_Y \to i^{-1}\mathcal{O}_X \to \mathcal{O}_Y \to 0$$

$\Rightarrow$

$$0 \to \underbrace{\mathcal{H}om_{i^{-1}\mathcal{O}_X}(\mathcal{O}_Y, i^{-1}\mathcal{N})}_{i^{\#}\mathcal{N}} \to i^{-1}\mathcal{N} \to \mathcal{H}om_{i^{-1}\mathcal{O}_X}(i^{-1}\mathcal{I}_Y, i^{-1}\mathcal{N})$$

$$\Rightarrow i^{\#}\mathcal{N} \cong \{ s \in i^{-1}\mathcal{N} \mid \exists f \in i^{-1}\mathcal{I}_Y : s \cdot f = 0 \}$$

Claim: (a) $\forall$ $\mathcal{M}$ coherent right $\mathcal{D}_Y$-mod $\mathcal{M}$: $i^{\#} i_+ \mathcal{M} \xrightarrow{\cong} \mathcal{M}$

(b) $\forall$ $\mathcal{M}$ coherent right $\mathcal{D}_X$-mod. $\mathcal{N}$: $\mathcal{N} \xrightarrow{\cong} i_+ i^{\#}\mathcal{N}$

Check both statements locally, $X = \operatorname{Spec} A$, coord. $(x_1, \dots, x_n)$, $I_Y = (x_{r+1}, \dots, x_n)$, $B = A/I$. $M = \Gamma(Y, \mathcal{M})$, $i_+ M = M[\partial_{r+1}, \dots, \partial_n]$

Proof of (a): $\{ s \in M[\partial_{r+1}, \dots, \partial_n] \mid \exists f \in (x_{r+1}, \dots, x_n) : s f = 0 \}$

$$\simeq M \otimes 1 \subset M[\partial_{r+1}, \dots, \partial_n]$$

$\ulcorner$ notice: $i \in \{r+1, \dots, n\}$: $(m \otimes \partial_i) \cdot x_i = (m \otimes 1) \cdot (\partial_i x_i)$

$= (m \otimes 1) \cdot x_i \partial_i + m \otimes 1 = m \cdot x_i \otimes \partial_i + m \otimes 1 = m \otimes 1 \neq 0$

however $(m \otimes \partial_i) x_i^2 = (m \otimes 1) \cdot x_i = m \cdot x_i \otimes 1 = 0$ $\lrcorner$

Proof of (b): Let $N = \Gamma(X, \mathcal{N})$ be right $D(A) = \Gamma(X, \mathcal{D}_X)$-module, with $Supp(N) \subset V(I)$, i.e.,

$\forall s \in N : \exists f \in I : s \cdot f^k = 0$ for some $k \in \mathbb{N}$

To show: $N = N_0 \otimes_K K[\partial_{r+1}, \ldots, \partial_n]$, where

$$N_0 = \{ s \in N \mid s \cdot I = 0 \}.$$

remark: Put $T_i := x_i \partial_i \in D(A)$, then we have

$$\partial_{r+1}^{\alpha_{r+1}} \cdot \ldots \cdot \partial_n^{\alpha_n} \cdot (T_i - \alpha_i) = T_i \cdot \partial_{r+1}^{\alpha_{r+1}} \cdot \ldots \cdot \partial_n^{\alpha_n}$$

hence, for all $s \in N_0$:

$$\left[s \otimes \partial_{r+1}^{\alpha_{r+1}} \cdot \ldots \cdot \partial_n^{\alpha_n} \right] \cdot (T_i - \alpha_i) = \overbrace{s x_i}^{=0} \otimes \partial_i \partial_{r+1}^{\alpha_{r+1}} \cdot \ldots \cdot \partial_n^{\alpha_n} = 0$$

moreover: $\forall \alpha$: $T_i (T_i - 1) \cdot \ldots \cdot (T_i - \alpha) = x_i^{\alpha+1} \partial_i^{\alpha+1}$

since $\forall s \in N : s \cdot I^k = 0$, we have $s \cdot x_i^{\alpha_i+1} \partial_i^{\alpha_i+1} = 0$

for $\alpha_i >> 0$, hence $s \cdot T_i \cdot (T_i - 1) \cdot \ldots \cdot (T_i - \alpha_i) = 0$

Hence $s = \sum_{\alpha \geq 0} s_\alpha$ with $s_\alpha \in \ker(T_i - \alpha)$ eigenspace

Moreover: $[T_i, T_j] = 0 \quad \forall i \neq j \implies$

$$N = \bigoplus_{\alpha_{r+1},\dots,\alpha_n \in \mathbb{N}} N_{\alpha_{r+1},\dots,\alpha_n}$$

where $N_{\alpha_{r+1},\dots,\alpha_n} = \bigcap_{i=r+1}^{n} \ker(T_i - \alpha_i)$

To show: - $N_{0,\dots,0} = N_0$

- $\exists$ iso of right $D(A)$-modules:

$$N_0[\partial_{r+1},\dots,\partial_n] \simeq \bigoplus_{\alpha_{r+1},\dots,\alpha_n \in \mathbb{N}} N_{\alpha_{r+1},\dots,\alpha_n}$$

For simplicity, consider only case $r = n-1$:

$\implies I = (x_n)$, $N = \bigoplus_{\alpha \in \mathbb{N}} N_\alpha$, $N_\alpha = \ker(x_n \partial_n - \alpha) \subset N$

$B = A/(x_n)$ and each N_α is $D(B)$-module since

for all $k \in \{1,\dots,n-1\}$, x_k, ∂_k commutes with $T_n = x_n \partial_{x_n}$

We have morphisms

$$N_{\alpha-1} \xleftarrow{\cdot x_n} N_\alpha \xrightarrow{\cdot \partial_n} N_{\alpha+1}$$

since for $s \in N_\alpha$, we have:

$$s\partial_n T_n = s\partial_n x_n \partial_n = (s\, x_n \partial_n + s)\partial_n$$

$$= s\cdot\alpha\cdot\partial_n + s\cdot\partial_n = s\cdot\partial_n\cdot(\alpha+1)$$

$$s x_n \cdot T_n = s x_n x_n \partial_n = s x_n(\partial_n x_n - 1)$$

$$= s x_n \partial_n x_n - s x_n = s\alpha x_n - s x_n = s x_n(\alpha-1)$$

By construction (essentially by the assumption that $\mathrm{supp}(N) \subset Y$): $N_e = 0 \ \forall e < 0$, hence:

$$N_0 = \ker(\cdot x_n \partial_n) = \ker(\cdot x_n) \text{ (also called } N_0 \text{ earlier)}$$

Ex: $\forall \alpha \geq 0$: $N_0 \to N_\alpha$, $s \mapsto s\cdot\partial_n^\alpha$ is iso of right $D(B)$-modules

- $\exists$ iso of right $D(A)$-mod's: $N_0[\partial_n] \to \bigoplus_{\alpha\geq 0} N_\alpha = N$ $\square$