Lectures on D-modules, part II

Reminder from last term: - $A_n(k)$ Weyl algebra in n variables. Linear homogeneous systems of PDE's $\Longleftrightarrow$ left modules over $A_n(k)$

globally: X smooth alg $/ k$ (char $k=0$)

$\mathcal{D}_X$: sheaf of alg. differential op's

$\mathcal{M}$ coherent $\mathcal{D}_X$-module: $\mathcal{M} \in QCoh(\mathcal{O}_X)$ together with left action $\mathcal{D}_X \times \mathcal{M} \to \mathcal{M}$.

$\mathrm{char}(\mathcal{M}) \subset T^*X$ involutive subset (Gabber) defined using good filtration $F_\bullet \mathcal{M}$, but independent!

Def.: $\mathcal{M}$ holonomic $:\Longleftrightarrow \dim(\mathrm{char}(\mathcal{M})) = n \Leftrightarrow \underset{\mathcal{M}\ \mathrm{hol.}}{\mathrm{char}\ \mathcal{M}}$ Lagrangian

dual $\mathcal{D}_X$-module: $\mathbb{D}\mathcal{M} := \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \overset{\downarrow}{=} \mathcal{H}om^\bullet_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$

direct image under closed immersion $i: Y \hookrightarrow X$:

$\mathcal{M}$ coherent right $\mathcal{D}_Y$-module, then put

$$i_+ \mathcal{M} = i_* \left(\mathcal{M} \otimes_{\mathcal{D}_Y} \mathcal{D}_{X\to Y} \right), \text{ where}$$

$$\mathcal{D}_{X\to Y} = \mathcal{O}_Y \otimes_{i^{-1}\mathcal{O}_X} i^{-1}\mathcal{D}_X \quad (\mathcal{D}_Y, i^{-1}\mathcal{D}_X)\text{-bimodule}$$

Kashiwara's equivalence: i_+ gives equivalence of categ. $\mathrm{Mod}_{coh}(\mathcal{D}_Y) \simeq \mathrm{Mod}^Y_{coh}(\mathcal{D}_X)$

$$\mathrm{Mod}^Y_{coh}(\mathcal{D}_X) := \{ \mathcal{M} \in \mathrm{Mod}_{coh}(\mathcal{D}_X) \mid \mathrm{supp}(\mathcal{M}) \subset Y \}$$

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aim now: for general map $f: X \to Y$ and for left $\mathcal{D}_Y$-module $\mathcal{N}$, define inverse image/pull-back $f^+\mathcal{N} \in \mathrm{Mod}(\mathcal{D}_X)$.

Basic idea: use tangent map

$$Tf: \Theta_X \longrightarrow f^*\Theta_Y, \quad v \longmapsto [\,g \mapsto v(g\circ f)\,]$$

$\Theta_X = \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_X)$, $g \in \mathcal{O}_Y$, $v \in \Theta_X$

Define $f^*\mathcal{N} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{N}$

(usual pull-back as $\mathcal{O}$-module), then we

have: $f^*\mathcal{N} \simeq \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{N} \simeq \underbrace{\left(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y\right)}_{\mathcal{D}_{X\to Y}} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}$

$\simeq \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}$. Recall that $\mathcal{D}_{X\to Y}$

is $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule $\rightsquigarrow f^*\mathcal{N}$ is left

$\mathcal{D}_X$-module. More concretely: $\forall h \in \mathcal{O}_X, u \in f^{-1}\mathcal{N}$,

$v \in \Theta_X$, we have $\theta(h\otimes u) = \theta(h)\otimes u + (Tf)(\theta)(1\otimes u)$

notice that since $(Tf)(\theta) \in f^*\Theta_Y \Rightarrow (Tf)(\theta)(1\otimes u) \in f^*\mathcal{N}$.

<u>Def</u>: $f: X \to Y$ map between smooth alg. varieties

$\mathcal{N} \in \mathrm{Mod}(\mathcal{D}_Y)$

$\rightsquigarrow f^+\mathcal{N} := Lf^*\mathcal{N} = \mathcal{D}_{X\to Y} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N} \in D^{b,\leq 0}(\mathcal{D}_X)$

<u>examples</u>: - let $\mathcal{E}$ be smooth, i.e. a VB + integrable connection $\nabla : \mathcal{E} \to \Omega^1_Y \otimes_{\mathcal{O}_Y} \mathcal{E}$. Recall that $\mathcal{E}$ is then left $\mathcal{D}_Y$-module + $\mathcal{O}_Y$-free, in particular, flat $\rightsquigarrow f^+\mathcal{E} = f^*\mathcal{E}$ with connection

$$f^*\nabla : f^*\mathcal{E} \to f^*\Omega^1_Y \otimes_{\mathcal{O}_X} f^*\mathcal{E} \xrightarrow{Tf^\vee} \Omega^1_X \otimes_{\mathcal{O}_X} f^*\mathcal{E}$$

- let $X = \mathbb{A}^1$, $A_1 = \Gamma(X, \mathcal{D}_X)$ Weyl algebra.

let $M = A_1/A_1 \cdot x$ left A_1-module

($\mathcal{M}$ the corresponding sheaf of $\mathcal{D}_{\mathbb{A}^1}$-modules)

$i : \{0\} \hookrightarrow \mathbb{A}^1$ embedding. $i^+\mathcal{M} = ?$

use $i^+\mathcal{M} = \mathcal{O}_{\{0\}} \overset{\mathbb{L}}{\otimes}_{k[x]} A_1/A_1x$ an consider free resolution $k[x] \xrightarrow{x} k[x]$ of $\mathcal{O}_{\{0\}} = \dfrac{k[x]}{(x)}$

$$\Rightarrow i^+\mathcal{M} = \underset{\deg = -1}{A_1/A_1x} \xrightarrow{x\cdot -} \underset{\deg = 0}{A_1/A_1x}$$ not A_1-lin.

<u>Claim:</u> $H^0(i^+\mathcal{M}) = 0$; $H^{-1}(i^+\mathcal{M}) \simeq k\cdot[1]$

notice: $A_1/A_1x = k[\partial]$

<u>H^0:</u> clearly, since $x\partial = \partial x - 1$, we have

$$[1] = [-x\partial] \in \operatorname{Im}(x\cdot)$$

inductively, one shows $[\partial^k] \in \operatorname{Im}(x\cdot)\ \forall k$

$\Rightarrow H^0(i^+\mathcal{M}) = 0$

<u>H^{-1}:</u> $[\partial^k] \in \ker(x\cdot) \Rightarrow [x\cdot\partial^k] = [P\cdot x]$

$\overset{\text{ex.}}{\Longleftrightarrow} k = 0$ $\square$

- consider again $i : \{0\} \hookrightarrow \mathbb{A}^1$ and $\mathcal{N} = \mathcal{D}_{\mathbb{A}^1}$

Since $\mathcal{D}_{\mathbb{A}^1}$ is $\mathcal{O}_{\mathbb{A}^1}$-flat $\leadsto i^+\mathcal{N} = i^*\mathcal{N} = \mathcal{D}_{\mathbb{A}^1}/x\cdot\mathcal{D}_{\mathbb{A}^1}$

$\simeq k[\partial]$ not k-finite

<u>Consequence</u>: in general, $f^+\mathcal{N}$ is not $\mathcal{D}_X$-coherent even if $\mathcal{N} \in \mathrm{Mod}_c(\mathcal{D}_Y)$

<u>aim: find conditions on $(f, \mathcal{N})$</u> s.t. coherence is preserved

Motivation: Cauchy-Kovalevskaya-Theorem

in dim $= 1$: let $F : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ be real analytic, then the initial value problem

$$\boxed{u' = F(u) \quad ; \quad u(0) = 0} \quad (*)$$

has a unique analytic solution near 0.

<u>Pf</u>: Taylor series Ansatz

$$u(t) = \sum_{n=0}^{\infty} u^{(n)}(0) \frac{t^n}{n!}$$

show that such u is convergent if $(*)$ holds we have $u' = F(u) \Rightarrow u^{(2)} = F'(u) \cdot u'$

$$= F'(u) \cdot F(u)$$

$$u^{(3)} = \left(F'(u) \cdot F(u)\right)' = F''(u) u' \cdot F(u) + F'(u)^2 \cdot u'$$

$$= F''(u) \cdot F(u)^2 + F'(u)^2 \cdot F(u)$$

$$\vdots$$

$$u^{(n)} = P_n\left(F(u), F'(u), \dots, F^{(n-1)}(u)\right)$$

for some universal $P_n \in \mathbb{Z}_{\geq 0}[x_0, \dots, x_{n-1}]$

(e.g. $P_1 = x_0$, $P_2 = x_0 \cdot x_1$, $P_3 = x_2 \cdot x_0^2 + x_1^2 \cdot x_0$, ...)

$$\Rightarrow |u^{(n)}(0)| \leq P_n\left(|F(0)|, |F^{(1)}(0)|, \dots, |F^{(n-1)}(0)|\right)$$

since coeff. of P_n in $\mathbb{Z}_{\geq 0}$ and $u(0) = 0$

<u>Key point</u>: Let $G : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be real analytic s.t. $|F^{(n)}(0)| \leq G^{(n)}(0)$ $\forall\, n \geq 0$, suppose further that v is <u>analytic</u> solution to

$$\boxed{v' = G(v) \ ; \ v(0) = 0} \quad (**)$$

Then since $v^{(n)}(0) = P_n\left(G(0), G^{(1)}(0), \dots, G^{(n-1)}(0)\right)$

and $|F^{(n)}(0)| \leq G^{(n)}(0)$ (and $P_n \in \mathbb{Z}_{\geq 0}[x]$)

$\Rightarrow |u^{(n)}(0)| \leq v^{(n)}(0) \Rightarrow u(t)$ is analytic

"G majorant of F"

Can we construct G: Write

$$F(t) = \sum_{n \geq 0} F^{(n)}(0) \frac{t^n}{n!} \quad \text{konvergent for } |t| < r$$

$$\Rightarrow \exists C > 0, \ |F^{(n)}(0)| \leq C \cdot \frac{n!}{r^n}$$

Put $G(t) := \sum_{n \geq 0} C \cdot \left(\frac{t}{r}\right)^n = C \cdot \frac{1}{1 - \frac{t}{r}} = C \frac{r}{r - t}$ on $|t| < r$

Then to solve $v' = G(v) = C \frac{r}{r - v}$, $v(0) = 0$

<u>separation of variables:</u> $\frac{dv}{dt} = c\frac{r}{r-v}$

$$\Leftrightarrow v\cdot dv - r\cdot dv = -c\cdot r\cdot dt \quad |\int$$

$$\frac{1}{2}v^2 - r\cdot v = -cr\cdot t$$

$$\Leftrightarrow v^2 - 2r\cdot v + 2crt = 0$$

$$v = r \pm \sqrt{r^2 - 2crt}$$

$$= r \pm r\sqrt{1 - \frac{2ct}{r}}$$

analytic for $|t| < \frac{r}{2c}$ $\square$

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generalization to higher dim. (i.e. PDE case):

consider $P = \sum_{|\alpha|\le k} f_\alpha \partial^\alpha \in \mathbb{R}\{x_1,\dots,x_n\}\langle\partial_1,\dots,\partial_n\rangle$

and equation $Pu = 0$ with initial value cond.

$$u_{|H} = g_0, \; (\partial_n u)_{|H} = g_1, \; \ldots \ldots \; (\partial_n^{(k-1)} u)_{|H} = g_{k-1}$$

where $H = \{x_n = 0\} \subset \mathbb{R}^n$, $g_i \in \mathbb{R}\{x_1, \ldots, x_{n-1}\}$.

Rewrite $Pu = 0$ as

$$f_{(0,\ldots,0,k)}(\underline{x}) \cdot \partial_n^k \cdot u = - \sum_{\underline{\alpha} : \alpha_n < k} f_\alpha \partial^\alpha \cdot u$$

From initial conditions, we know all functions $(\partial^\alpha \cdot u)_{|H}$ $\forall \underline{\alpha} : \alpha_n < k$, this leads to

<u>Def</u>: P is said to be non-characteristic w.r.t $H = \{x_n = 0\} \iff f_{(0,\ldots,0,k)}(x_1, \ldots, x_{n-1}, 0) \neq 0$ $\forall (x_1, \ldots, x_{n-1}) \in H$

<u>Clear</u>: P non-char. w.r.t. $H \Longrightarrow$ from $Pu = 0$ & $(\partial_n^{(l)} u)_{|H} = g_l$ we can compute all $\partial^\alpha u_{|H}$,

hence in particular all $(\partial^\alpha u)(0)$ $\rightsquigarrow$ Taylor series uniquely determined.

Th (Cauchy-Kovalevskaya): P as above, $\deg(P) \le k$, non-char. w.r.t. $H = \{x_n = 0\}$. Then the boundary value problem

$$\boxed{Pu = 0 \,, \quad (\partial_n^{(\ell)} u)_{|H} = g_\ell \quad \forall \ell \le k-1}$$

has a unique analytic solution near $0 \in \mathbb{R}^n$