Introduction to V-filtrations

aim: If $\mathcal{M} \in \mathrm{Mod}_{rh}(\mathcal{D}_X)$, $K = DR^{\cdot}(\mathcal{M}) \in \mathrm{Perv}(X)$, $t \in \mathcal{O}_X$ (smooth) $\Rightarrow \psi_t K^{\cdot}, \varphi_t K^{\cdot} \in \mathrm{Perv}(X_0)$, where $X_0 = \{t=0\}$. Question: What is $DR^{-1}(\psi_t K^{\cdot})$, $DR^{-1}(\varphi_t K^{\cdot})$?

Def.: Let $Y \subset X$ be a smooth hypersurface $\mathcal{I}_Y = \mathcal{I}(Y) \subset \mathcal{O}_X$, put $\forall k \in \mathbb{Z}$:

$$V_k \mathcal{D}_X = \left\{ P \in \mathcal{D}_X \,\middle|\, P(\mathcal{I}_Y^j) \subset \mathcal{I}_Y^{j-k} \right\}$$

with $\mathcal{I}^{\ell} = \mathcal{O}_X \quad \forall \ell \leq 0$

If $(t, x_2, \dots, x_n) \in X$ is local coord. system with $Y = \{t=0\} \Rightarrow P \in V_0 \mathcal{D}_X \Leftrightarrow P = \sum a_{\underline{\alpha}, j} \cdot \partial_{\underline{x}}^{\underline{\alpha}} \cdot (t\partial_t)^j$

with $a_{\underline{\alpha}, j} \in \mathcal{O}_X$ and $\forall k \in \mathbb{N}$

$P \in V_{-k} \mathcal{D}_X$: $P = t^k \cdot Q$, $Q \in V_0 \mathcal{D}_X$

$P \in V_k \mathcal{D}_X$: $P = \sum_{0 \le \ell \le k} Q_\ell \partial_t^\ell$, $Q_\ell \in V_0 \mathcal{D}_X$

Properties: - $V_k \mathcal{D}_X \cdot V_\ell \mathcal{D}_X \subset V_{k+\ell} \mathcal{D}_X$

with „=" for $k, \ell \le 0$ or $k, \ell \ge 0$

- $V_0 \mathcal{D}_X$ sheaf of rings, $V_k \mathcal{D}_X$ is locally free $V_0 \mathcal{D}_X$-module

- $\bigcup_k V_k \mathcal{D}_X = \mathcal{D}_X$; $(V_k \mathcal{D}_X)_{|X \setminus Y} \cong \mathcal{D}_{X \setminus Y}$

rk: - $gr_0^V \mathcal{D}_X$ is sheaf of rings as well

- class of $[t\partial_t] \in gr_0^V \mathcal{D}_X$ is independent of coordinate system

- locally, we have $gr_0^V \mathcal{D}_X = \mathcal{D}_Y [t\partial_t]$

good V-filtrations: Put $R_V \mathcal{D}_X := \bigoplus_{k \in \mathbb{Z}} V_k \mathcal{D}_X \cdot \tau^k$

Rees ring. Then for $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$, let

$U_\bullet(\mathcal{M})$ be increasing filtration by $V_0 \mathcal{D}_X$-

submodules s.t. $V_k \mathcal{D}_X \cdot U_\ell \mathcal{M} \subseteq U_{\ell+k} \mathcal{M}$.

Prop: The following are equivalent

1.) $R_U \mathcal{M} := \bigoplus_{k \in \mathbb{Z}} U_k \mathcal{M} \cdot \tau^k$ is $R_V \mathcal{D}_X$-coherent

2.) All $U_k \mathcal{M}$ are $V_0 \mathcal{D}_X$-coherent and

locally $\exists\, k_0 \in \mathbb{N}$ s.t. $\forall\, k \in \mathbb{N}$:

$V_k \mathcal{D}_X \cdot U_{k_0} \mathcal{M} = U_{k+k_0} \mathcal{M}$ & $V_{-k} \mathcal{D}_X \cdot U_{-k_0} \mathcal{M} = U_{-k_0-k} \mathcal{M}$

3.) locally, $\exists$ epimorphism

$$\bigoplus_{i=1}^{p} \mathcal{D}_X \cdot e_i \xrightarrow{\phi} \mathcal{M} \to 0 \quad \text{and } a_1, \dots, a_p \in \mathbb{Z}$$

s.t. $U_k \mathcal{M} = \left\{ \sum \phi\left(V_{k+a_i} \mathcal{D}_X \cdot e_i \right) \right\}$

Def: $U_\bullet \mathcal{M}$ as above is called good V-filt.

Rk: $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ in $\mathrm{Mod}_{h}(\mathcal{D}_X)$

Then good V-filt of $\mathcal{M}_2$ induces good V-filt of $\mathcal{M}_1$ and $\mathcal{M}_3$

Pf: - for $\mathcal{M}_3$, use 3.) above

- for $\mathcal{M}_1$, use 1.) above

and $0 \to R_U \mathcal{M}_1 \to R_U \mathcal{M}_2 \to R_U \mathcal{M}_3 \to 0$

Lemma: U, U' good V-filt on $\mathcal{M}$

$\Rightarrow$ locally $\exists\, k_1, k_2$:

$$U_{\bullet + k_1}\mathcal{M} \subset U'_{\bullet}\mathcal{M} \subset U_{\bullet + k_2}\mathcal{M}$$

Proof: exercise (manipulation of indices)

In the sequel, let E be any lift to $V_0\mathcal{D}_X$ of the class $[t\partial_t] \in gr_0^V \mathcal{D}_X$

<u>Prop</u>: $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$. Then

1.) $\exists$ good V-filtration $U_\bullet \mathcal{M}$

and $\exists$ locally $b(s) \overset{\neq 0}{\in} \mathbb{C}[s]$ s.t.

$$b(E+k)\, U_k \mathcal{M} \subset U_{k-1} \mathcal{M} \quad \forall k \in \mathbb{Z}$$

2.) Same statement for any good V-filtration $U_\bullet \mathcal{M}$

3.) Let locally $\mathcal{M} = \sum_{i=1}^{\ell} \mathcal{D}_X \cdot m_i$, then

$$\exists\, b(s) \neq 0 \in \mathbb{C}[s] : b(E)\, m_i \in \sum_{j=1}^{\ell} V_{-1}\mathcal{D}_X \cdot m_j$$

<u>Proof</u>: 1.) $\Rightarrow$ 2.): Let $U_\bullet \mathcal{M}$ be good V-filt satisfying 1.) with B-pol $b(s)$, let $U'_\bullet \mathcal{M}$ be any good V-filt. By previous lemma, $\exists k_1, k_2$:

$$U_{\bullet + k_1} \mathcal{M} \subset U'_\bullet \mathcal{M} \subset U_{\bullet + k_2} \mathcal{M}$$

Moreover $\forall k \in \mathbb{Z}$:

$$b(E+k+k_1)\cdot \ldots \cdot b(E+k+k_2-1)\cdot b(E+k+k_2)\,\underbrace{U'_k M}_{\text{by Lemma } \subset U_{k+k_2}M} \subset U'_{k-1}M$$

$$\subset U_{k+k_2-1}M$$

$$\subset U_{k+k_2-2}M$$

$$\subset U_{k+k_1-1}M \overset{\text{by Lemma}}{\subset} U'_{k-1}M$$

hence $U'_\cdot M$ has good V-filtration

2.) $\Rightarrow$ 3.) After earlier proposition,

$U_kM := \sum V_k D_X \cdot m_i$ is good, hence,

after 2.) $b(E)\cdot U_0 M \subset \underbrace{U_{-1}M}_{=\sum V_{-1}D_X m_i}$, i.e.

$b(E)\cdot \overset{\in}{m_i} \subset \; = \sum V_{-1} D_X m_i$

3.) $\Rightarrow$ 1.) Put $U_kM := \sum V_k D_X \cdot m_i$

we have (ex.): $b(E+k)\cdot V_k D_X \subset V_k D_X \cdot b(E) + V_{k-1} D_X$

hence $b(E+k)\cdot \mathcal{U}_k \mathcal{M} \subset \sum V_k \mathcal{D}_X \cdot V_{-1}\mathcal{D}_X \cdot m_i$

$$+ \sum V_{k-1} \mathcal{D}_X m_i$$

$$= \sum V_{k-1} \mathcal{D}_X \cdot m_i = \mathcal{U}_{k-1} \mathcal{M} \qquad \square$$

<u>Def.</u>: $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$ is called specializable (along $\{t=0\}$) iff it satisfies cond' of the previous proposition.

Let $\mathcal{U}_\bullet \mathcal{M}$ be good V-filt. of such $\mathcal{M}$,

Then $\{ b(s) \mid b(E+k)\mathcal{U}_k \subset \mathcal{U}_{k-1} \ \forall k \}$ is ideal in $\mathbb{C}[s]$, we call monic generator $b^{\mathcal{U}_\bullet \mathcal{M}}(s)$ the local B.-S.-polynomial of $\mathcal{U}_\bullet \mathcal{M}$.

<u>Rk</u>: $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$

$\mathcal{M}_2$ specializable $\Longleftrightarrow$ $\mathcal{M}_1, \mathcal{M}_3$ specializable

<u>Theorem</u>: Let $\sigma : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ be a section of $\pi : \mathbb{C} \to \mathbb{C}/\mathbb{Z}$, let $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$ be specializable. $\exists!$ good V-filtration $V^\sigma_\bullet \mathcal{M}$ s.t. its B.-S.-pol has roots in $\mathrm{Im}(\sigma)$.

<u>Proof</u>: Let $U_\bullet \mathcal{M}$ be any good V-filtr. with B.S.-pol. $b^{U_\bullet}(s)$. We can assume:

$\forall \alpha$ root of $b^{U_\bullet}(s)$: $\mathrm{Re}(\alpha) \leq \mathrm{Re}\, \sigma\pi(\alpha)$

(by shift of $U_\bullet \mathcal{M}$).

Let $\lambda \in \mathbb{C}$ with: $b^{U_\bullet}(\lambda) = 0$ & $\mathrm{Re}\,\lambda < \mathrm{Re}\,\sigma\pi(\lambda)$, write $b^{U_\bullet}(s) = (s-\lambda)^\ell \cdot b_1(s)$ $\quad b_1(\lambda) \neq 0$

Put $\boxed{U'_k \mathcal{M} := U_{k-1}\mathcal{M} + (E+k-\lambda)^\ell U_k \mathcal{M}}$

<u>Claim</u>: $U'_\bullet \mathcal{M}$ is good with B.S.-pol $(s-\lambda-1)^\ell b_1(s)$

<u>Pf of claim</u>: $(E+k-\lambda-1)^{\ell} b_1(E+k)\, \mathcal{U}'_k \mathcal{M} =$

$$(E+k-\lambda-1)^{\ell} \cdot b_1(E+k)\, \mathcal{U}_{k-1}\mathcal{M} + (E+k-\lambda-1)^{\ell} \underbrace{\underbrace{b_1(E+k)(E+k-\lambda)}_{=b^{u}(E+k)}{}^{\ell}\, \mathcal{U}_k \mathcal{M}}_{\subset \mathcal{U}_{k-1}\mathcal{M}}$$

$$= b_1(E+k)\cdot \underbrace{(E+k-\lambda-1)^{\ell}\, \mathcal{U}_{k-1}\mathcal{M}}_{\subset \mathcal{U}'_{k-1}\mathcal{M}} + \underbrace{(E+k-\lambda-1)^{\ell}\, \mathcal{U}_{k-1}\mathcal{M}}_{\subset \mathcal{U}'_{k-1}\mathcal{M}}$$

$$\subset \mathcal{U}'_{k-1}\mathcal{M} \qquad \square$$

Hence, inductively, we can construct $V^{\sigma}\mathcal{M}$ s.t. $\forall$ α root of $b^{V^{\sigma}}(s)$: $\operatorname{Re}\alpha \geq \operatorname{Re}\sigma\pi\alpha$, i.e. with roots of $b^{V^{\sigma}}(s) \subset \operatorname{Im}(\sigma)$

<u>remains to show:</u> $V^{\sigma}\mathcal{M}$ is unique with this property: let $\mathcal{U}.\mathcal{M}$ and $V.\mathcal{M}$ be two such filtrations, show $\mathcal{U}.\mathcal{M} \subset V.\mathcal{M}$

we know: $\exists\, \ell \in \mathbb{Z}$: $\mathcal{U}_k \mathcal{M} \overset{(*)}{\subset} \mathcal{V}_{k+\ell}\mathcal{M} \ \forall k$.

$\ell \leq 0 \Rightarrow \mathcal{U}_\bullet \mathcal{M} \subset \mathcal{V}_\bullet \mathcal{M}$

$\ell \in \mathbb{N} \setminus \{0\}$: we have $\gcd\left(b^{\mathcal{U}_\bullet}(s+k), b^{\mathcal{V}_\bullet}(s+k+\ell)\right) = 1$,

since roots of $b^{\mathcal{U}}(s)$ and $b^{\mathcal{V}}(s)$ are both in $\operatorname{Im}(\sigma)$

$\Rightarrow \exists\, p, q \in \mathbb{C}[s]$: $p(s) \cdot b^{\mathcal{U}_\bullet}(s+k) + q(s) \cdot b^{\mathcal{V}_\bullet}(s+k+\ell) = 1$

hence, $\forall m \in \mathcal{U}_k \mathcal{M}$:

$$m = p(E) \cdot \underbrace{b^{\mathcal{U}_\bullet}(E+k)m}_{\in\, \mathcal{U}_{k-1}\mathcal{M}} + q(E) \cdot \underbrace{b^{\mathcal{V}_\bullet}(E+k+\ell)m}_{\in\, \mathcal{V}_{k+\ell-1}\mathcal{M}}$$

$$\in \underbrace{\mathcal{U}_{k-1}\mathcal{M}}_{\overset{(*)}{\subset}\, \mathcal{V}_{k+\ell-1}\mathcal{M}} + \mathcal{V}_{k+\ell-1}\mathcal{M} \subset \mathcal{V}_{k+\ell-1}\mathcal{M}$$

hence we have shown: If $\ell \in \mathbb{N}_{>0}$ is such that

$\mathcal{U}_k \mathcal{M} \subset \mathcal{V}_{k+\ell}\mathcal{M}$, then $\mathcal{U}_k \mathcal{M} \subset \mathcal{V}_{k+\ell-1}\mathcal{M}$

by induction $\mathcal{U}_\bullet \mathcal{M} \subset \mathcal{V}_\bullet \mathcal{M}$, symmetry „$\supset$" $\square$