Introduction to V-filtrations II

recall from last time: $Y = \{t=0\} \subset X$ smooth

$V_k \mathcal{D}_X = \{P \in \mathcal{D}_X \mid P(\mathcal{I}_Y^j) \subset \mathcal{I}_Y^{j-k}\}$

$U_\bullet M$ good V-filt: $R_U M$ is $R_V \mathcal{D}_X$-coherent

M specializable $\underset{\text{Df.}}{\Longleftrightarrow}$ $\exists$ good V-filt $U_\bullet M$ and $0 \neq b^U \in \mathbb{C}[s]$: $\forall k \in \mathbb{Z}$: $b^U(E+k)\, U_k M \subset U_{k-1} M$.

<u>Theorem</u>: $\sigma: \mathbb{C}/\mathbb{Z} \to \mathbb{Z}$ section of $\pi: \mathbb{C} \to \mathbb{C}/\mathbb{Z}$

$\leadsto \exists!$ good V-filt. $U_\bullet M$ s.t. roots of $b^U \subset \operatorname{Im} \sigma$.

<u>aim</u>: refine a good V-filtration so as to include the roots of its B.-S.-pol. to the data.

<u>Def.</u>: M specializable, $m \in M$, $b_m(s)$ minimal monic polynomial s.t. locally $b_m(E) \cdot m \in V_{-1}\mathcal{D}_X m = V_0 \mathcal{D}_X \cdot t \cdot m$ is called B.-S.-pol. of m.

write $\operatorname{ord}_Y(m) := \{\alpha \in \mathbb{C} \mid b_m(\alpha) = 0\}$

<u>notation</u>: $\alpha \in \mathbb{C}$, $\sigma_\alpha : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ splitting with image $\{z \in \mathbb{C} \mid -\alpha - 1 \leq \operatorname{Re}(z) < -\alpha\}$

write $b_\alpha(s)$ for $b^{V^{\sigma_\alpha}}(s)$.

From now on, choose order of $\mathbb{C}$ extending order of $\mathbb{R}$.

<u>Def</u>: $\mathcal{M}$ specializable, define $\forall x \in X, \alpha \in \mathbb{C}$:

$$V_\alpha \mathcal{M}_x := \left\{ m \in \mathcal{M}_x \mid \operatorname{ord}_y(m) \subset [-\alpha-1, \infty) + i\mathbb{R} \right\}$$

Then $(V_\bullet \mathcal{M})$ is called canonical V-filtration of $\mathcal{M}$

<u>Lemma</u>: $\forall \alpha \in \mathbb{C}$, the restriction $(V_\beta \mathcal{M})_{\beta \in \alpha + \mathbb{Z}}$ is a good V-filtration and we have

$$\forall k \in \mathbb{Z}: \ V_{\alpha+k} \mathcal{M} = V_k^{\sigma_\alpha} \mathcal{M}$$

<u>Proof</u>: By definition $V_\bullet^{\sigma_\alpha} \mathcal{M}$ is good V-filt. with B.-S.-pol $b^\alpha(s)$ $(\operatorname{ord} b^\alpha(s) \subset \operatorname{Im} \sigma_\alpha)$

hence, $\forall \ell \in \mathbb{Z}$: $b_\alpha(E+k+\ell) V_{k+\ell}^{\sigma_\alpha} \mathcal{M} \subset V_{k+\ell-1}^{\sigma_\alpha} \mathcal{M}$

$\Rightarrow b^\alpha(s+\ell)$ is B.-S.-pol of shifted good V-filt. $V^{\sigma_\alpha}_{\bullet+\ell}\mathcal{M}$

<u>ex.:</u> We have $V^{\sigma_\alpha}_{\bullet+\ell}\mathcal{M} \cong V^{\sigma_{\alpha+\ell}}_{\bullet}\mathcal{M}$

in particular, for $\bullet=0$, we get

$$V^{\sigma_\alpha}_{k}\mathcal{M} \simeq V^{\sigma_{\alpha+k}}_{0}\mathcal{M}$$

Hence we only need to show: $V_\alpha\mathcal{M} \cong V^{\sigma_\alpha}_0\mathcal{M}$

<u>"$\subset$":</u> $m \in V_\alpha\mathcal{M}$, let $k \in \mathbb{Z}_{>0}$

recall that $b_m(E)\cdot m \in t\cdot V_0\mathcal{D}_X\cdot m$

$\Rightarrow b_m(E-1)\cdot t\cdot V_0\mathcal{D}_X m \in t\cdot b_m(E)\cdot V_0\mathcal{D}_X m$

$$\subset t V_0\mathcal{D}_X\cdot b_m(E)m \subset t^2V_0\mathcal{D}_X\cdot m \subset V_{-2}\mathcal{D}_X m$$

$\Rightarrow \ldots \Rightarrow b_m(E-k+1)\cdot\ldots\cdot b_m(E)m \in V_{-k}\mathcal{D}_X\cdot m$

$k \gg 0 \Rightarrow m \in V^{\sigma_\alpha}_k\mathcal{M} \Rightarrow b_m(E-k+1)\cdot\ldots\cdot b_m(E)m \in V^{\sigma_\alpha}_0\mathcal{M}$

<u>by def:</u> $b^\alpha(E+k)m \in V^{\sigma_\alpha}_{k-1}\mathcal{M}$

and $\gcd\big(b^\alpha(s+k), \overbrace{b_m(s-k-1)\cdot\ldots\cdot b_m(s)}^{=:B(s)}\big) = 1$:

we have: - roots of $b^\alpha(s) \subset [-\alpha-1,-\alpha) + i\mathbb{R}$

- roots of $b^\alpha(s+k) \subset [-\alpha-k-1, -\alpha-k) + i\mathbb{R}$

- since $m \in V_\alpha \mathcal{M} \Rightarrow$ roots of $b_m(s) \subset [-\alpha-1, \infty)$

$\Rightarrow$ roots of $B(s) \subset [-\alpha-1, \infty)$

$\Rightarrow \exists p, q \in \mathbb{C}[s] \quad 1 = p(s) \cdot b^\alpha(s+k) + q(s) \cdot B(s)$

$$\Rightarrow m = p(E) \cdot \underbrace{b^\alpha(E+k) \cdot m}_{\in V^{G_\alpha}_{k-1}\mathcal{M}} + q(E) \cdot \underbrace{B(E) \cdot m}_{\in V^{G_\alpha}_0 \mathcal{M}}$$

hence we have shown: $k>0$ s.t. $m \in V^{G_\alpha}_k \mathcal{M} \Rightarrow m \in V^{G_\alpha}_{k-1}\mathcal{M}$

iterate (works for all $k>0$) $\Rightarrow m \in V^{G_\alpha}_0 \mathcal{M}$ ✓

„$\supset$": $m \in V^{G_\alpha}_0 \mathcal{M}$. We have $V^{G_\alpha}_\bullet \mathcal{M} \cap \mathcal{D}_X m = V^{G_\alpha}_\bullet(\mathcal{D}_X m)$

(check functional equation that defines $V^{G_\alpha}_\bullet(\mathcal{D}_X m)$),

its B.-S.-pol is divisor of $b^\alpha(s)$

$(V_\bullet \mathcal{D}_X) \cdot m$ is good V-filt $\Rightarrow \exists \kappa : V^{G_\alpha}_{-\kappa}(\mathcal{D}_X m) \subset V_{-1}\mathcal{D}_X m$

as above: $b^\alpha(E-\kappa+1) \cdot \ldots \cdot b^\alpha(E) \cdot m \in V^{G_\alpha}_{-\kappa}(\mathcal{D}_M) \Rightarrow b^\alpha(E-\kappa+1) \cdot \ldots \cdot b(E)\, m \in V_{-1}\mathcal{D}_X m$

$\Rightarrow b_m(s) \mid b^\alpha(s-\kappa+1) \cdot \ldots \cdot b^\alpha(s) \Rightarrow \mathrm{ord}(m) \subset [-\alpha-1, \infty) \Rightarrow m \in V_\alpha \mathcal{M}_D$

<u>examples:</u> 1.) $(\mathcal{E}, \nabla) \to \Delta^*_t$ flat holom. bundle

e: $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\} \xrightarrow{e^z} \Delta^*_t$

and put $V := \ker(e^*\nabla_{\partial_z} : e^*\mathcal{E} \circlearrowleft)$.

we have $M = M_{ss} \cdot M_u \in \mathrm{Aut}(V)$ and

$V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$, $V_\lambda = \ker(M_{ss} - \lambda \mathrm{Id}_V)$. Then

$\forall \alpha$ s.t. $e^{2\pi i \alpha} = \lambda$, $s \in V_\lambda$, put

$$es(\alpha, s)(z) := e^{-z(\alpha \mathrm{Id} + N)} s(z), \text{ where } N = \frac{1}{2\pi i} \log M_u$$

$$\Rightarrow es(\alpha, s)(z + 2\pi i) = e^{-(z+2\pi i)(\alpha \mathrm{Id} + N)} \underbrace{s(z+2\pi i)}_{(M \cdot s)(z)}$$

$$= e^{-z(\alpha \mathrm{Id} + N)} \cdot e^{-2\pi i \alpha} \cdot e^{2\pi i N} M \cdot s(z)$$

$$= e^{-z(\alpha \mathrm{Id} + N)} e^{-\log M_u} \cdot M_u \cdot e^{-2\pi i \alpha} \cdot M_{ss} \cdot s(z)$$

$$= e^{-z(\alpha \mathrm{Id} + N)} \cdot \lambda^{-1} M_{ss} s(z) \overset{s \in V_\lambda}{=} es(\alpha, s)(z).$$

hence $es(\alpha,s)(e^z) =: es(\alpha,s)(t) \in \Gamma(\Delta^*, \mathcal{E})$.

ex: $t \cdot es(\alpha,s) = es(\alpha+1, s)$

$$\nabla_z e^* es(\alpha,s) = (t\partial_t) es(\alpha,s) = -(\alpha \mathrm{Id} + N) es(\alpha,s)$$

chose basis $v_1, \dots, v_k$ of V, put, $\forall \alpha : e^{2\pi i \alpha} = \lambda$.

$C_\alpha := \bigoplus_{i=1}^{k} \mathbb{C}\, es(\alpha, v_i)$ and

$V_\alpha \mathcal{E} := \bigoplus_{\beta \in (\alpha-1,\alpha]} \mathcal{O}_{\mathbb{C}} \cdot C_\alpha$, $\quad V_{<\alpha} \mathcal{E} := \bigoplus_{\beta \in [\alpha-1,\alpha)} \mathcal{O}_{\mathbb{C}} \cdot C_\alpha$

$V_{<\infty} \mathcal{E} = \bigcup_\alpha V_\alpha$, then

$V_{<\infty}\mathcal{E}$ is $\mathcal{D}_{\mathbb{C}}$ module extending $(\mathcal{E}, \nabla)$

$j_+ j^+ V_{<\infty} \mathcal{E} \simeq V_{<\infty} \mathcal{E}$

$V_\bullet \mathcal{E}$ is canonical V-filtration on $V_{<\infty}\mathcal{E}$.

<u>Def.</u>: $\mathcal{M}$ specializable along $\{t=0\}=Y$

Put $V_{<\alpha}\mathcal{M} := \bigcup\limits_{\beta<\alpha} V_\beta\mathcal{M}$ and

$$gr_\alpha\mathcal{M} := V_\alpha\mathcal{M} / V_{<\alpha}\mathcal{M}$$

<u>Lemma</u>: $\mathcal{M}$ specializable along Y

1.) $\forall \alpha\in\mathbb{C}$, $gr^V_\alpha\mathcal{M}$ is $gr^V_0\mathcal{D}_X$-coherent,

$E+\alpha+1$ is nilpotent endomorphism of $gr^V_\alpha\mathcal{M}$ with $(E+\alpha+1)^{\mathrm{mult}(-\alpha-1,\, b^V(s))}=0$

2.) $\mathrm{Re}\,\alpha\in(k-1,k] \rightsquigarrow gr_\alpha\mathcal{M}$ is generalized eigenspace of E on $V_k\mathcal{M}/V_{k-1}\mathcal{M}$.

3.) $\forall k\in\mathbb{Z}$: $\exists$ iso

$$V_k\mathcal{M}/V_{k-1}\mathcal{M} \simeq \bigoplus_{\alpha\in(k-1,k]} gr_\alpha\mathcal{M}$$

Corollary: $\forall \alpha \in \mathbb{C}$: $\mathrm{gr}_\alpha \mathcal{M}_{|Y}$ is $\mathcal{D}_Y$-coherent

Pf: we have seen: $\mathrm{gr}_0^V \mathcal{D}_X \cong \mathcal{D}_Y[E]$ locally

$\mathrm{gr}_\alpha \mathcal{M}$ is $\mathrm{gr}_0^V \mathcal{D}_X$-coherent, but E is locally nilpotent. $\square$

Prop: $\mathcal{M}$ specializable along Y

1.) $\forall \alpha \geq 0, \forall k \in \mathbb{N}$: $V_{\alpha+k}\mathcal{M} \cong V_k(\mathcal{D}_X) \cdot V_\alpha \mathcal{M}$

2.) $\forall \alpha > 0, \forall k \in \mathbb{N}$: $V_{-\alpha-k}\mathcal{M} \cong V_{-k}(\mathcal{D}_X) V_{-\alpha}\mathcal{M}$

Proof: 1.) from previous lemma, part 1.) $\Longrightarrow$

minimal pol. of $E \in \mathrm{End}(V_{\alpha+k}\mathcal{M}/V_{\alpha+k-1}\mathcal{M})$ is

$b^{\alpha+k}(s)$ (roots in $[-\alpha-1-k, -\alpha-k)$)

$\Longrightarrow \mathrm{ggT}(s+1, b^{\alpha+k}(s)) = 1 \quad \forall k \geq 1$

$\Rightarrow \exists p(s), q(s)$: $p(s)\cdot(s+1) + q(s)\, b^{\alpha+k}(s) = 1$

hence, for all $m \in V_{\alpha+k}\mathcal{M}$:

$$p(E)\cdot(E+1)m + q(E)\cdot \underbrace{b^{\alpha+k}(E)\cdot m}_{\in V_{\alpha+k-1}\mathcal{M}} = m$$

$\Rightarrow$ on $V_{\alpha+k}\mathcal{M} / V_{\alpha+k-1}\mathcal{M}$:

$$p(E)\cdot(E+1)[m] = [m]$$

$\Rightarrow E+1 = t\partial_t + 1 = \partial_t t$ is invertible

on $V_{\alpha+k}\mathcal{M} / V_{\alpha+k-1}\mathcal{M}$

<u>exercise</u>: this implies

$$V_{\alpha+k}\mathcal{M} = V_k\mathcal{D}_X \cdot V_\alpha \mathcal{M} \quad \forall k \geq 0$$

2.) similarly

$\square$