Representability and the YONEDA-Lemma.

Def. a) Let $F: \mathcal{C} \to Set$ be a (co- or contravariant) functor with $\mathcal{C}$ a locally small category.
A *representation* for F is a selection of an object $c \in \mathcal{C}$ with a specified natural isomorphism
$$\mathcal{C}(c,-) \cong F, \text{ if } F \text{ is covariant, or}$$
$$\mathcal{C}(-,c) \cong F, \text{ if } F \text{ is contravariant.}$$
If such a c exist, then we say that F is *representable* and *represented by* c.

b) Let $\mathcal{C}$ be a category and $c \in \mathcal{C}$ an object. We say that c is
- *initial* if the constant functor $*: \mathcal{C} \to Set$ that sends every object to the singleton set is represented by c.
- *terminal* if the constant functor $*: \mathcal{C}^{op} \to Set$ is represented by c.

remark. $\cdot$ c is initial $\Leftrightarrow$ $\mathcal{C}(c,a)$ is a singleton set for all $a \in \mathcal{C}$.
$\cdot$ c is final $\Leftrightarrow$ $\mathcal{C}(a,c)$ —〃—.

Examples. a) $1_{Set}: Set \to Set$ (identity functor) is represented by the singleton set 1: $Set(1,X) = X^1 \cong X$ for every set X.
Let $f: X \to Y$ be a morphism (= map) of sets:

$$\begin{tikzcd} Set(1,X) \arrow[r, "\cong"] \arrow[d, "f_*"'] & X \arrow[d, "f"] \\ Set(1,Y) \arrow[r, "\cong"] & Y \end{tikzcd} \qquad \begin{tikzcd} (1 \mapsto x) \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[d, mapsto] \\ (1 \mapsto f(x)) \arrow[r, mapsto] & f(x) \end{tikzcd}$$

b) $U: Grp \to Set$ (forgetful functor) is represented by $\mathbb{Z} \in Grp$.
$\varphi: G \to H$ group homomorphism:

$$\begin{tikzcd} Grp(\mathbb{Z},G) \arrow[r, "\cong"] \arrow[d, "\varphi_*"'] & UG \arrow[d, "\varphi"] \\ Grp(\mathbb{Z},H) \arrow[r, "\cong"] & UH \end{tikzcd} \qquad \begin{tikzcd} (1 \mapsto g) \arrow[r, mapsto] \arrow[d, mapsto] & g \arrow[d, mapsto] \\ (1 \mapsto \varphi(g)) \arrow[r, mapsto] & \varphi(g) \end{tikzcd}$$

(this is an isomorphism because every homomorphism $\mathbb{Z} \to G$ is determined by the image of $1 \in \mathbb{Z}$.)

c) R unital ring. $U: Mod_R \to Set$ is represented by $R \in Mod_R$.
$\varphi: M \to N$ R-module homomorphism:

$$\begin{tikzcd}
Mod_R(R,M) \arrow[r, "\cong"] \arrow[d, "\varphi_*"'] & UM \arrow[d, "\varphi"] \\
Mod_R(R,N) \arrow[r, "\cong"] & UN
\end{tikzcd}
\qquad
\begin{tikzcd}
(1 \mapsto m) \arrow[r, mapsto] \arrow[d, mapsto] & m \arrow[d, mapsto] \\
(1 \mapsto \varphi(m)) \arrow[r, mapsto] & \varphi(m)
\end{tikzcd}$$

d) Path: $Top \to Set$ is represented by $I = [0,1] \subseteq \mathbb{R}$,
Loop: $Top_* \to Set$ —"— S^1 (with fixed base point)
by definition: $Top(I,X) \cong \text{Path}\,X$ $\forall X \in Top$,
$Top_*(S^1,Y) \cong \text{Loop}\,X$ $\forall Y \in Top_*$.

e) (CURRYing; contravariant example)
$Set(- \times A, B): Set^{op} \to Set$ is represented by $B^A = Set(A,B)$
$Set(X \times A, B) \cong Set(X, B^A)$, $\forall X \in Set$
$(\varphi: X \times A, B) \mapsto (\varphi^c: X \to B^A,\ x \mapsto (\varphi^c_x: A \to B,\ a \mapsto \varphi(x,a)))$
(Inverse: $(\psi: X \to B^A) \mapsto (\psi_c: X \times A \to B,\ (x,a) \mapsto [\psi(x)](a))$)

f) Let $N \trianglelefteq G$ be a normal subgroup of the group G and
$\pi: G \twoheadrightarrow G/N$ the canonical epimorphism.
$F: Grp \to Set$
$H \mapsto \{\varphi \in Grp(G,H) \mid N \leq \operatorname{Ker}\varphi\} \subseteq Grp(G,H)$
$(\psi: H \to H') \mapsto (\psi_*: FH \to FH',\ \varphi \mapsto \psi \circ \varphi)$
where $\operatorname{Ker}\psi\circ\varphi \geq \operatorname{Ker}\varphi \geq N$ because for $g \in \operatorname{Ker}\varphi$
$\psi\circ\varphi(g) = \psi(e_H) = e_{H'}$.
Universal property of quotient group:
F is represented by $G/N \in Grp$.

„old version"
$\varphi: G \to H$ Group hom.
$N \leq \operatorname{Ker}\varphi$
$\Rightarrow \exists!\, \bar\varphi: G/N \to H$:
$\varphi = \bar\varphi \circ \pi$

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & H \\
G/N \arrow[ur, dashed, "\exists!\,\bar\varphi"'] &
\end{tikzcd}$$

If $\varphi: H \to H'$ is a group homomorphism, we obtain

$$\begin{tikzcd}
Grp(G/N, H) \arrow[r, "\cong"] \arrow[d, "\varphi_*"'] & FH \arrow[d, "\varphi_*"] \\
Grp(G/N, H') \arrow[r, "\cong"] & FH'
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde\psi: G/N \to H,\ [g] \mapsto \psi(g) \arrow[r, mapsto] \arrow[d, mapsto] & \psi: G \to H,\ g \mapsto \psi(g) \arrow[d, mapsto] \\
\varphi\circ\tilde\psi: G/N \to H',\ [g] \mapsto \varphi\circ\psi(g) \arrow[r, mapsto] & \varphi\circ\psi: G \to H',\ g \mapsto \varphi\circ\psi(g)
\end{tikzcd}$$

given by $(\tilde\psi: G/N \to H) \mapsto (\psi = \tilde\psi \circ \pi: G \to H)$.
($\operatorname{Ker}\tilde\psi \geq \operatorname{Ker}\pi = N$ by definition)
The inverse is $\psi \in FH \longmapsto (\tilde\psi: G/N \to H, [g] \to \psi(g))$,
which is well defined as $[g] = [h] \Leftrightarrow gh^{-1} \in N \leq \operatorname{Ker}\psi$,
so $\psi(g)\psi(h)^{-1} = 1 \Rightarrow \psi(g) = \psi(h)$.

Problem. Given a functor $F: C \to Set$, what information is necessary to define a natural isomorphism $\alpha: C(c,-) \cong F$ - or more generally a natural transformation $\alpha: C(c,-) \Rightarrow F$?

Example. ordinal category ω $(0 \to 1 \to 2 \to \_ \to n \to \_)$.
$F: \omega \to Set$ is a family $(F_n)_{n\in\omega}$ of sets with functions $f_{n,n+1}: F_n \to F_{n+1}$, $n \in \omega$. $\omega(k,-): \omega \to Set$, represented by $k \in \omega$, maps $n \in \omega$ to the empty set, if $n<k$, or to a singleton set $*$, else.
If $\alpha: \omega(k,-) \Rightarrow F$ is a natural transformation, we obtain:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "\alpha_0"'] & \emptyset \arrow[r] \arrow[d, "\alpha_1"'] & _ \arrow[r] & \emptyset \arrow[r] \arrow[d, "\alpha_{k-1}"'] & * \arrow[r] \arrow[d, "\alpha_k"'] & * \arrow[r] \arrow[d, "\alpha_{k+1}"'] & _ \\
F_0 \arrow[r, "f_{0,1}"'] & F_1 \arrow[r] & _ \arrow[r] & F_{k-1} \arrow[r, "f_{k-1,k}"'] & F_k \arrow[r, "f_{k,k+1}"'] & F_{k+1} \arrow[r] & _
\end{tikzcd}$$

the α_n, $n<k$, carry no information and the α_n, $n \geq k$, are equivalent to a single element from each F_n which we will also call α_n. Then naturality implies

$$\alpha_{n+1} = f_{n,n+1}(\alpha_n) = f_{n,n+1}(f_{n-1,n}(\_(f_{k+1,k}(\alpha_k))\_)) \quad \forall n \geq k.$$

Thus α is completely determined by $\alpha_k = \alpha_k(1_k) \in F_k$ with $1_k \in \omega(k,k)$.

Theorem. (YONEDA lemma)
Let C be a locally small category and $F: C \to Set$ a functor. Then, for every $c \in C$ there is a bijection

$$Hom(C(c,-), F) \cong Fc,$$

that associates $\alpha: C(c,-) \Rightarrow F$ to $\alpha_c(1_c) \in Fc$.
Furthermore, this correspondence is natural in c and F.
This also implies that $Hom(C(c,-), F)$ is a set.

Proof. Obviously,

$$\Phi: Hom(C(c,-), F) \to Fc, \quad \alpha \mapsto \Phi(\alpha) := \alpha_c(1_c)$$

is a function. We now need to find an inverse for it:
Let $x \in Fc$ be arbitrary. We now desire a natural transformation $\Psi(x): C(c,-) \Rightarrow F$, i.e. for every $f \in C(c,d)$ we want, that

the square
$$\begin{tikzcd} C(c,c) \arrow[r, "\Psi(x)_c"] \arrow[d, "f_*"'] & Fc \arrow[d, "Ff"] \\ C(c,d) \arrow[r, "\Psi(x)_d"'] & Fd \end{tikzcd}$$
commutes. For $1_c \in C(c,c)$, this implies
$$\Psi(x)_d(f) = Ff(\Psi(x)_c(1_c)).$$
As Ψ is supposed to be an inverse of Φ, we conclude
$$\Phi(\Psi(x)) = \Psi(x)_c(1_c) \overset{!}{=} x \in Fc$$
Thus we define
$$\Psi : Fc \to \mathrm{Hom}(C(c,-),F), \quad \Psi(x)_d(f) := Ff(x).$$
Now let $g: d \to e$ be an arbitrary morphism in C. We want the following square to commute:
$$\begin{tikzcd} C(c,d) \arrow[r, "\Psi(x)_d"] \arrow[d, "g_*"'] & Fd \arrow[d, "Fg"] \\ C(c,e) \arrow[r, "\Psi(x)_e"'] & Fe \end{tikzcd}$$
For $f \in C(c,d)$ arbitrary, we obtain:
$$\Psi(x)_e \circ g_*(f) = \Psi(x)_e(g \circ f) = F(g \circ f)(x) \overset{F\text{ functor}}{=} Fg \circ Ff(x)$$
$$= Fg(\Psi(x)_d(f)) = Fg \circ \Psi(x)_d(f)$$
and as such $\Psi(x)$ is natural and Ψ is well-defined.
Lastly we want to show that Ψ is a left inverse of Φ (right invertability: $\Phi(\Psi(x)) = \Psi(x)_c(1_c) = x$ by definition).
Let $\alpha : C(c,-) \Rightarrow F$ be a natural transformation. Then:
$$\Psi(\Phi(\alpha)) = \Psi(\alpha_c(1_c)).$$
By the commutativity of the square
$$\begin{tikzcd} C(c,c) \arrow[r, "\alpha_c"] \arrow[d, "f_*"'] & Fc \arrow[d, "Ff"] \\ C(c,d) \arrow[r, "\alpha_d"'] & Fd \end{tikzcd} \qquad (*)$$
we obtain
$$\Psi(\Phi(\alpha))_d(f) = Ff(\alpha_c(1_c)) = \alpha_d(f)$$
and thus $\Psi(\Phi(\alpha)) = \alpha$.
Naturality in F:
Given a natural transformation $\beta : F \Rightarrow G$, we want to show that the corresponding element to $\beta \circ \alpha : C(c,-) \Rightarrow F \Rightarrow G$ in Gc is the image of that corresponding to $\alpha : C(c,-) \Rightarrow F$

in Fc under $\beta_c : Fc \to Gc$:

$$\begin{tikzcd}
\mathrm{Hom}(C(c,-),F) \arrow[r, "\cong", "\Phi_F"'] \arrow[d, "\beta_*"'] & Fc \arrow[d, "\beta_c"] \\
\mathrm{Hom}(C(c,-),G) \arrow[r, "\cong", "\Phi_G"'] & Gc
\end{tikzcd}$$

This diagram obviously commutes as:

$$\Phi_G(\beta\circ\alpha) = (\beta\circ\alpha)_c(1_c) = \beta_c(\alpha(1_c)) = \beta_c(\Phi_F(\alpha))$$

Similarly, given a morphism $f: c \to d$ in C, we want to show that the corresponding element to $\alpha\circ f^*: C(d,-) \Rightarrow C(c,-) \Rightarrow F$ in Fd is the image of the corresponding element to $\alpha: C(c,-) \Rightarrow F$ in Fc under $Ff: Fc \to Fd$:

$$\begin{tikzcd}
\mathrm{Hom}(C(c,-),F) \arrow[r, "\cong", "\Phi_c"'] \arrow[d, "(f^*)^*"'] & Fc \arrow[d, "Ff"] \\
\mathrm{Hom}(C(d,-),F) \arrow[r, "\cong", "\Phi_d"'] & Fd
\end{tikzcd}$$

We can see:

$$Ff(\Phi_c(\alpha)) = Ff(\alpha_c(1_c)) \overset{!}{=} (\alpha\circ f^*)_d(1_c) = \Phi_d((f^*)^*\circ\alpha),$$

as

$$\begin{aligned} C(d,d) &\xrightarrow{f^*} C(c,d) \xrightarrow{\alpha_d} Fd, \\ 1_d &\longmapsto f \longmapsto \alpha_d(f) \end{aligned}$$

and $(*)$ implies

$$(\alpha\circ f^*)_d(1_c) = \alpha_d(f) = Ff(\alpha_c(1_c))$$

as desired. $\square$

Remark. • $(c,F) \in C\times Set^C$, where Set^C is the category of functors $C \to Set$ with naturals transformations between them as morphisms. $ev: C\times Set^C \to Set$, $(c,F) \mapsto Fc$ is a bifunctor

• $C^{op} \xrightarrow{y} Set^C$, $\quad c \mapsto C(c,-)$, $f\uparrow \;\mapsto\; \uparrow f^*$, $d \mapsto C(d,-)$

possible size issue here

$$\mathrm{Hom}(y(-),-): C\times Set^C \xrightarrow{y\times 1_{Set^C}} (Set^C)^{op}\times Set \xrightarrow{\mathrm{Hom}} Set$$
$$(c,F) \longmapsto (C(c,-),F) \longmapsto \mathrm{Hom}(C(c,-),F)$$

• YONEDA lemma is equivalent to

$$\begin{tikzcd}
C\times Set^C \arrow[r, bend left=50, "\mathrm{Hom}(y(-),-)", ""{name=U, below}] \arrow[r, bend right=50, "ev"', ""{name=D, above}] & Set
\arrow[from=U, to=D, Rightarrow, "\Phi"', "\cong"]
\end{tikzcd}$$