Analytical Tools for Quantum Many-Body Physics

- General motivation:
  - Full quantum mechanical understanding of many-body systems can be essential to capture their basic behavior

    ↳ Prominent examples: Complex macro-molecules in quantum chemistry, high T_c superconductors, ...
  - Solution of complex many-body problems may enable revolutionary applications, such as room-temperature superconductors, efficient fertilizers, quantum supremacy in information processing.

- Fundamental caveat: Exponential complexity of generic quantum many-body (QMB) systems in their number of constituents.

  - If a single particle state is defined in a Hilbert space h of dimension d, a state of N particles is defined in the ((anti)-symmetrized sub-space of the) space $\mathcal{H} = \underbrace{h \otimes h \otimes \cdots \otimes h}_{N\text{-factors}}$ of dimension d^N.

- This course is about analytical tools for taming this complexity to (at least approximately) solve broad classes of QMB problems.

— Outline —

0: Recap: Noninteracting particles, 2nd Quantization, observables in free systems, Green's functions

A: Perturbative Methods (and beyond)

A.1: Zero-temperature perturbation theory

A.2: Matsubara Green's function technique for finite T

A.3: Schwinger-Keldysh formalism for non-equilibrium systems

B: Bosonization and Luttinger Liquid theory

B.1: Breakdown of perturbation theory

B.2: Constructive Bosonization

B.3: Luttinger Liquids: Physical properties and correlation functions

B.4: Impurity problems and non-equilibrium transport.

Outlook: Not in this lecture.

End of Outline

Recap: Practical calculation of observables in free systems

- One-body observables (in systems of N identical particles)

$$O = \sum_{i=1}^{N} \mathbb{1} \otimes \cdots \otimes \underset{\substack{\uparrow \\ \text{pos. } i}}{\mathcal{O}} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \equiv \sum_{i=1}^{N} \mathcal{O}^{(i)} \overset{\substack{\text{2nd Quant.} \\ \downarrow}}{\underset{\substack{\uparrow \\ \text{see supplementary slides}}}{=}} \sum_{\alpha,\beta} O_{\alpha\beta}\, a_\alpha^\dagger a_\beta \,.$$

with $O_{\alpha\beta} = \langle \alpha | \mathcal{O} | \beta \rangle$.

single part. states

- Fermions: System in Slater determinant state

$$|\Psi\rangle = S_-|\alpha_1, \dots, \alpha_N\rangle$$

$$\boxed{\langle \Psi | O | \Psi \rangle = \sum_{i=1}^{N} \langle \alpha_i | \mathcal{O} | \alpha_i \rangle}\,,$$

since in $\langle\Psi|O|\Psi\rangle = \frac{1}{N!}\sum_{P,P'}(-1)^{|P|+|P'|}\underbrace{\langle\alpha_1,\dots,\alpha_N|P^\dagger O P'|\alpha_1,\dots,\alpha_N\rangle}_{=0 \text{ unless } P=P'}$.

— Bosons: System in state $|\Psi\rangle = \frac{S_+}{\sqrt{n_1!\cdots}}|\alpha_1,\dots,\alpha_N\rangle$

$\rightsquigarrow \langle\Psi|O|\Psi\rangle = \sum_{i=1}^{N}\langle\alpha_i|O|\alpha_i\rangle$ as long as $\langle\alpha_i|\alpha_j\rangle \in \{0,1\}$.

[More complicated situation for Bosons occupying arbitrary non-orthogonal states! Example: 2 particles in states $|\alpha\rangle$ and $|\gamma\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + |\beta\rangle)$ with $\langle\alpha|\beta\rangle = 0$. Then,

$\langle\Psi|O|\Psi\rangle = \frac{1}{3}(5O_{\alpha\alpha} + 2O_{\alpha\beta} + 2O_{\beta\alpha} + O_{\beta\beta}) \neq \langle\alpha|O|\alpha\rangle + \langle\gamma|O|\gamma\rangle$]