- For the full GF of an interacting system with $H = H_0 + V$, this equation of motion reads as

$$i\hbar\, \partial_t\, G^r(k, t-t') = \delta(t-t') - i\,\theta(t-t') \langle \{ [a_k, H_0 + V](t), a_k^\dagger(t') \} \rangle$$

$$\Leftrightarrow \left(i\hbar\, \partial_t - \varepsilon_k\right) G^r(k, t-t') = \delta(t-t') - i\,\theta(t-t') \underbrace{\langle \{ [a_k, V](t), a_k^\dagger(t') \} \rangle}_{\Gamma(t-t')}$$

- The additional term $\Gamma(t-t')$ is quartic in the $a_k$-operators and thus represents a two-particle correlation function. This is a direct consequence of the non-linear Heisenberg equations of motion for $a_k(t)$.

- Foundation of perturbation theory: The $2n$-particle correlation functions appearing at $n$-th order in $V$ as <u>free</u> expectation values, can be factorized into products of $n$ single particle GFs using Wick's theorem.

# A.1.a Basic concepts of zero temperature perturbation theory

- At $T=0$, equilibrium expectation values are to be evaluated in the ground state $|E_0\rangle$ of the system (with $H|E_0\rangle = E_0|E_0\rangle$).

↳ Fundamental problem: For $H = H_0 + V$, $|E_0\rangle$ is in general not known.

↳ Solution: Switch on interactions adiabatically between $t=-\infty$ and $t=0$, and off between $t=0$ and $t=+\infty$.

$$\boxed{V \to V_\alpha(t) = Ve^{-\alpha|t|}, \quad \alpha > 0}$$

⇝ Then, $\lim_{t\to\pm\infty} H(t) = H_0$, and $|E_0\rangle$ may be adiabatically connected to $|0\rangle$ (with $H_0|0\rangle = E_0^0|0\rangle$) during this procedure.

Recap: Interaction picture (Dirac picture) for $H = H_0 + V_t$

possible explicit time-dep.

- $$|\Psi_D(t)\rangle := U_0^{-1}(t,t_0)\,|\Psi_S(t)\rangle = U_0^{-1}(t,t_0)\,U_S(t,t')\,|\Psi_S(t')\rangle =$$

$$= \underbrace{U_0^{-1}(t,t_0)\,U_S(t,t')\,U_0(t',t_0)}_{=:\,U_D(t,t')}\,|\Psi_D(t')\rangle$$

- Generic operators $O_D(t) = e^{iH_0(t-t_0)}\,O_S\,e^{-iH_0(t-t_0)}$

- Equations of motion: $i\hbar\,|\dot\Psi_D(t)\rangle = \dot U_0^\dagger(t,t_0)\,|\Psi_S(t)\rangle + U^\dagger(t,t_0)\,|\dot\Psi_S(t)\rangle$

$$= i\left(U_0^\dagger(t,t_0)\,H_0 - U_0^\dagger(t,t_0)\,H\right)|\Psi_S(t)\rangle = i\,\underbrace{U_0^\dagger\,(-V_t)\,U_0(t,t_0)}_{V_t^D(t)}\,|\Psi_D(t)\rangle$$

$$\Leftrightarrow \quad \boxed{i\frac{d}{dt}|\Psi_D(t)\rangle = V_t^D(t)\,|\Psi_D(t)\rangle}$$

and equivalently $i\frac{d}{dt} U_D(t,t') = V_t^D(t)\, U_D(t,t')$

$$\rightsquigarrow \boxed{U_D(t,t') = T e^{-i\int_{t'}^{t} d\tau\, V_\tau^D(\tau)}},$$

or, for the density matrix $\dot{\rho}_D(t) = i\,[\rho_D, V_t^D](t)$.

[ • From now on, we will by default work in the Dirac rep. ]

• Gell-Mann-Low theorem for adiabatic switching:

If $\lim_{\alpha\to 0} \frac{U_\alpha^D(0,-\infty)|0\rangle}{\langle 0|U_\alpha^D(0,-\infty)|0\rangle} = \lim_{\alpha\to 0} \frac{|\Psi_\alpha^D(0)\rangle}{\langle 0|\Psi_\alpha^D(0)\rangle}$ exists in every order perturbation theory of $U_\alpha^D(0,-\infty) = T e^{-i\int_{-\infty}^{0} d\tau\, V_\alpha^D(\tau)}$, it is an eigenstate of $H$, and thus the ground state of $H$ in the absence of level crossings during the adiabatic switching.

• Causal Green's function $G_c(x,t,x',t') = -i\langle T(\Psi(x,t)\Psi^+(x',t'))\rangle =$

$$= -i\theta(t-t')\langle E_0|\Psi_H(x,t)\,\Psi_H^+(x',t')|E_0\rangle \;\overset{\text{bosons}}{\underset{\text{fermions}}{\pm}}\; (-i)\theta(t'-t)\langle E_0|\Psi_H^+(x',t')\,\Psi_H(x,t)|E_0\rangle .$$

- Goal: Use adiabatic switching to simplify $\langle E_0|\dots|E_0\rangle$ expectation values.

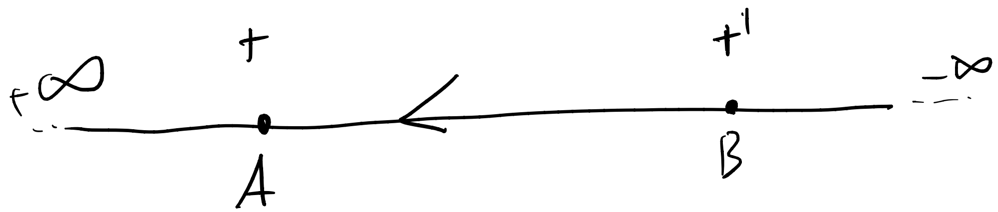
$$\langle E_0|\,O^H(t)\,|E_0\rangle \overset{\text{Gell-Mann-Low}}{=} \frac{\langle 0|U^D(\infty,t)\,O^D(t)\,U^D(t,-\infty)|0\rangle}{\langle 0|\underbrace{U^D(+\infty,-\infty)}_{=:\,S}|0\rangle}.$$

$$\boxed{\frac{U^D(0,\infty)|0\rangle}{\langle 0|U^D(0,\infty)|0\rangle} = \frac{U^D(0,-\infty)|0\rangle}{\langle 0|U^D(0,-\infty)|0\rangle} = |\tilde{E}_0\rangle}$$

$$\boxed{|E_0\rangle = \frac{|\tilde{E}_0\rangle}{\sqrt{\langle \tilde{E}_0|\tilde{E}_0\rangle}}}$$

- Two operators: $\langle E_0|A^H(t)\,B^H(t')|E_0\rangle = \dfrac{\langle 0|U^D(\infty,t)A^D(t)\,U^D(t,t')B^D(t')\,\overset{\times|0\rangle}{U(t',-\infty)}}{\langle 0|S|0\rangle}$

Graphically for $t > t'$



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- Back to causal GF:

$$i\,G_c(x,t;x',t') = \langle E_0 | T(\psi(x,t)\,\psi^+(x',t')) | E_0\rangle =$$

$$= \frac{\langle 0 | T\left(e^{-i\int_{-\infty}^{+\infty} d\tau\, V_\tau^D(\tau)}\,\psi^D(x,t)\,\psi^{D+}(x',t')\right) | 0\rangle}{\langle 0 | S | 0\rangle}.$$

⇝ Explicitly: $i\,G^c(x,t;x',t') = \frac{1}{\langle 0|S|0\rangle} \sum_{\nu=0}^{\infty} \frac{1}{\nu!}(-i)^\nu \int_{-\infty}^{+\infty} dt_1 \dots \int_{-\infty}^{+\infty} dt_\nu \times$

$$\langle 0 | T\left(V_{t_1}^D(t_1)\,V_{t_2}^D(t_2) \dots V_{t_\nu}^D(t_\nu)\,\psi^D(x,t)\,\psi^{D+}(x',t')\right) | 0\rangle$$

⇝ $\nu$-th order terms contain $\langle 0|\dots|0\rangle$ expectation values of $(4\nu+2)$ field operators (assuming quartic $V$). How to simplify these expressions? ⇝ Answer: Wick's theorem.

- Wick's theorem: (Dicussion for Fermions (harder case))

(i) Normal ordering and Fermi vacua

- Annihilation operators with respect to a vacuum defined by $\gamma_\lambda|0\rangle = 0$.
- Example Fermi sphere (spin neglected)

$$\left.\begin{array}{ll} a_k^+|0\rangle = 0 & \forall |k|<k_F \\ a_k|0\rangle = 0 & \forall |k|>k_F \end{array}\right\} \Rightarrow \gamma_k = \begin{cases} a_k^+ , & |k|<k_F \\ a_k , & |k|>k_F \end{cases}.$$

- Normal ordering for $\gamma_k$'s simply means that all $\gamma_k^+$'s stand to the left of $\gamma_k$'s.

↳ Examples: $N(\gamma_{k_1}\gamma_{k_2}^+\gamma_{k_3}) = (-1)\gamma_{k_2}^+\gamma_{k_1}\gamma_{k_3} = \gamma_{k_2}^+\gamma_{k_3}\gamma_{k_1} =$

$$= N(\gamma_{k_2}^+\gamma_{k_3}\gamma_{k_1}) = -N(\gamma_{k_3}\gamma_{k_2}^+\gamma_{k_1}).$$

↳ Every permutation of neighboring Fermi operators gives a "− sign".

Often $N(\dots)$ is denoted by $:\dots:$.