- Importantly $\langle 0 | N(\dots) | 0 \rangle = 0$

  ↳ arbitrary strings of field operators!

(ii) Def. Contraction: $\underbracket{A(t)\; B(t')} \equiv T\left(A(t)\,B(t')\right) - N\left(A(t)\,B(t')\right)$.

⇝ Contractions of only $\gamma$'s or only $\gamma^{+}$'s give $0$.

- Remaining nontrivial cases:

$$\underbracket{\gamma_k(t)\;\gamma^{+}_{k'}(t')} = \begin{cases} \gamma_k(t)\gamma^{+}_{k'}(t') + \gamma^{+}_{k'}(t')\gamma_k(t) & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases}$$

$$\underbracket{\gamma^{+}_{k'}(t')\;\gamma_k(t)} = \begin{cases} -\gamma_k(t)\gamma^{+}_{k'}(t') - \gamma^{+}_{k'}(t')\gamma_k(t) & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases}$$

- Assuming free time-evolution $\gamma_k(t) = e^{\pm i \varepsilon_k t} \gamma_k$, all contractions are 0 unless $k = k'$.

- Express explicitly in terms of physical $d$'s instead of $\gamma$'s

↳ $|k| > k_F$ :
$(a = \gamma)$
(outside Fermi-sphere)

$$a_k(t)\, a^\dagger_{k'}(t') = \delta_{k,k'} \begin{cases} e^{-i(\varepsilon_k - \mu)(t-t')} & t > t' \\ 0 & t < 0 \end{cases}$$

$$d^\dagger_{k'}(t')\, d_k(t) = \delta_{k,k'} \begin{cases} -e^{-i(\varepsilon_k - \mu)(t-t')} & t > t' \\ 0 & t < t' \end{cases}$$

↳ $|k| < k_F$
$(a = \gamma^\dagger)$
(inside Fermi-sphere)

$$d_k(t)\, a^\dagger_{k'}(t') = \delta_{k,k'} \begin{cases} 0 & t > t' \\ -e^{-i(\varepsilon_k - \mu)(t-t')} & t < t' \end{cases}$$

$$d^\dagger_{k'}(t')\, d_k(t) = \delta_{k,k'} \begin{cases} 0 & t > t' \\ e^{-i(\varepsilon_k - \mu)(t-t')} & t < t' \end{cases}$$

- Compare to

$$i\,G_c^0(k,t-t') = \Big(\theta(t-t')\big(1-\langle n_k\rangle_0\big) - \theta(t'-t)\langle n_k\rangle_0\Big)e^{-i(\varepsilon_k-\mu)(t-t')}$$

$$\left[\langle a_k a_k^\dagger\rangle_0 = 1-\langle \underbrace{a_k^\dagger a_k}_{n_k}\rangle_0\right]$$

$$\boxed{\begin{aligned} \underbrace{a_k(t)\,a_k^\dagger(t')} &= i\,G_c^0(k,t-t') \\ \underbrace{a_k^\dagger(t')\,a_k(t)} &= -i\,G_c^0(k,t-t') \end{aligned}}$$

- Subtle point for $t=t'$, where $G_c^0$ is not well defined. This is important for contractions within one physical operator (e.g. interaction $V$).

$\rightsquigarrow$ Convention: Since $T(\ldots)$ leaves such operators in their natural order (i.e. creation operators to the left), we define the "hair splitting" of time: $t_{\text{anihilation}} - t_{\text{creation}} \equiv 0^-$.

$$\rightsquigarrow \underbracket{a_k(t)\, a_k^\dagger(t)} = i G_c^0(k, 0^-), \quad \underbracket{a_k^\dagger(t)\, a_k(t)} = -i G_c^0(k, 0^-)$$

with $i G_c^0(k, 0^-) = -\langle n_k \rangle_0$.

(iii) Statement of Wick's theorem as an operator identity

$$T(ABCD) = N(ABCD) + N(\underbracket{AB}CD) + N(\underbracket{ABC}D) +$$

$$N(\underbracket{ABCD}) + N(A\underbracket{BC}D) + N(A\underbracket{BCD}) +$$

$$N(AB\underbracket{CD}) +$$

$$+ \underbrace{N(\underbracket{AB}\,\underbracket{CD}) + N(ABCD) + N(\overbracket{A\underbracket{BC}D})}_{\equiv \text{ total pairing}}$$

in general arbitrary # of operators

(Combinatorial statement, proof by induction in # of operators)

- For our purposes: $\langle 0 | T(ABCD\dots) | 0 \rangle =$ total pairing

↳ products of free causal Green's functions!

$$\boxed{\langle 0 | N(\dots) | 0 \rangle = 0}$$

something non-contracted

- Goal of diagrammatic perturbation theory: Bring order into the factorially big number of total pairing terms.

# A.1.b: Feynman Diagrams

- Goal: Devise a systematic graphical representation of terms in the perturbative expansion of $G_c$.

  ↳ Details of resulting "Feynman rules" will depend on chosen model system, but the construction principles are generic.

- Paradigmatic example: Spin-independent electron-electron interactions such as screened Coulomb interaction (spin neglected).

$$\hat{W}(t) = \frac{1}{2}\int d^3x_1\, d^3x_2\; \Psi^\dagger(x_1,t)\,\Psi^\dagger(x_2,t)\; U(x_1-x_2)\; \Psi(x_2,t)\,\Psi(x_1,t).$$

- Shorthand notation $\quad U(1-2) \equiv U(x_1-x_2)\,\delta(t_1-t_2)$ ,

$$\Psi_1 = \Psi(\underbrace{x_1, t_1, \ldots}_{\text{spin etc.}}) \quad , \quad \int d1 \ldots = \int_{-\infty}^{+\infty} dt_1\, d^3x_1 \ldots \; ;$$

$$\rightsquigarrow \; iG_c(r,r') \overset{+\mathcal{O}(U^2)}{\approx} \frac{\overbrace{\langle 0|T(\psi_r\psi^\dagger_{r'})|0\rangle}^{= iG^0_c(r,r')} + (-i)\frac{1}{2}\int d1\int d2\, U(1-2)\langle 0|T(\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1\psi_r\psi^\dagger_{r'})|0\rangle}{1 + (-i)\frac{1}{2}\int d1\int d2\, U(1-2)\langle 0|T(\psi_1^\dagger\psi_2^\dagger\psi_2\psi_1)|0\rangle}$$

external variables
of GF