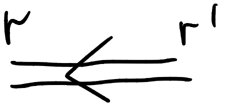
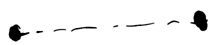
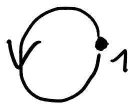
$$iG_c(r,r') \underset{+\mathcal{O}(U^2)}{\approx} \frac{iG_c^0(r,r') + (-i)\frac{1}{2}\int d1 \int d2\, U(1-2) \langle 0 | T(\psi_1^+ \psi_2^+ \psi_2 \psi_1 \psi_r \psi_{r'}^+) | 0 \rangle}{1 + (-i)\frac{1}{2}\int d1 \int d2\, U(1-2) \langle 0 | T(\psi_1^+ \psi_2^+ \psi_2 \psi_1) | 0 \rangle}$$

(i): $$\langle 0 | T(\psi_1^+ \psi_2^+ \psi_2 \psi_1 \psi_r \psi_{r'}^+) | 0 \rangle \overset{\text{Wick}}{=} \overbracket{\psi_1^+ \psi_1}\ \overbracket{\psi_2^+ \psi_2}\ \overbracket{\psi_r \psi_{r'}^+} + (-1)\, \overbracket{\psi_1^+ \psi_2}\ \overbracket{\psi_2^+ \psi_1}\ \overbracket{\psi_r \psi_{r'}^+} +$$
$$(-1)\, \overbracket{\psi_1^+ \psi_r}\ \overbracket{\psi_1 \psi_2^+}\ \overbracket{\psi_2 \psi_{r'}^+} + \overbracket{\psi_1^+ \psi_2}\ \overbracket{\psi_2^+ \psi_r}\ \overbracket{\psi_1 \psi_{r'}^+} +$$
$$(-1)\, \overbracket{\psi_1^+ \psi_r}\ \overbracket{\psi_2^+ \psi_2}\ \overbracket{\psi_1 \psi_{r'}^+} + (-1)\, \overbracket{\psi_1^+ \psi_1}\ \overbracket{\psi_2^+ \psi_r}\ \overbracket{\psi_2 \psi_{r'}^+}.$$

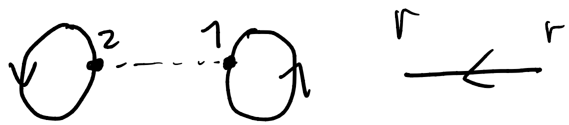
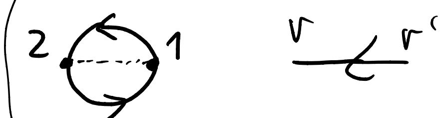
(ii): $$\langle 0 | T(\psi_1^+ \psi_2^+ \psi_2 \psi_1) | 0 \rangle \overset{\text{Wick}}{=} \overbracket{\psi_1^+ \psi_1}\ \overbracket{\psi_2^+ \psi_2} + (-1)\, \overbracket{\psi_1^+ \psi_2}\ \overbracket{\psi_2^+ \psi_1}.$$

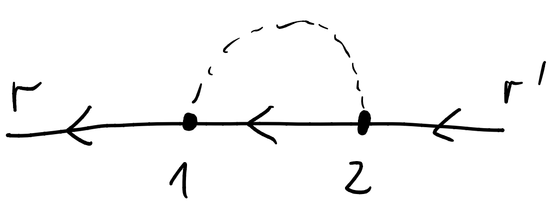
(vacuum amplitude)

- Elementary building blocks of Feynman diagrams for our model:

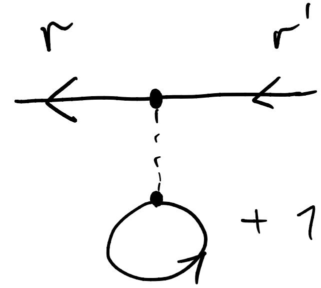
	$i\,G_c(r,r')$	(Full propagator (causal GF))
	$i\,G_c^0(r,r')$	(Free causal GF)
	$-i\,U(1-2)$	(Interaction potential)
	$\langle n\rangle_0 = \psi_1^+\psi_1$ (contraction)	(Free density)

(i) Graphical representation of $-\frac{i}{2}\int d1 \int d2\, U(1-2)\,\langle 0|T(\psi_1^+\psi_2^+\psi_2\psi_1\psi_r\psi_{r'}^+)|0\rangle$

" = " $\Big($  $\Big) - \Big($  $\Big) +$



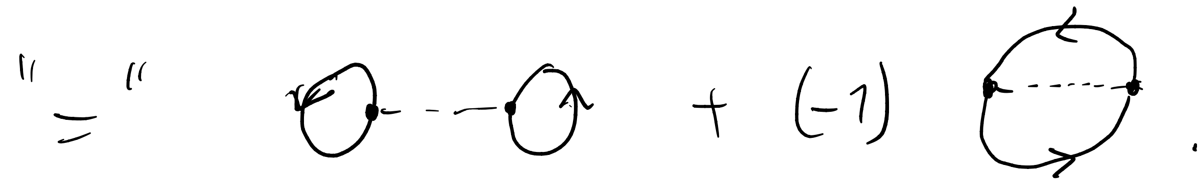
$+ 1 \leftrightarrow 2$

$+$ 

$+ 1 \leftrightarrow 2$

(only 4 of 6 diagrams are topologically distinct)

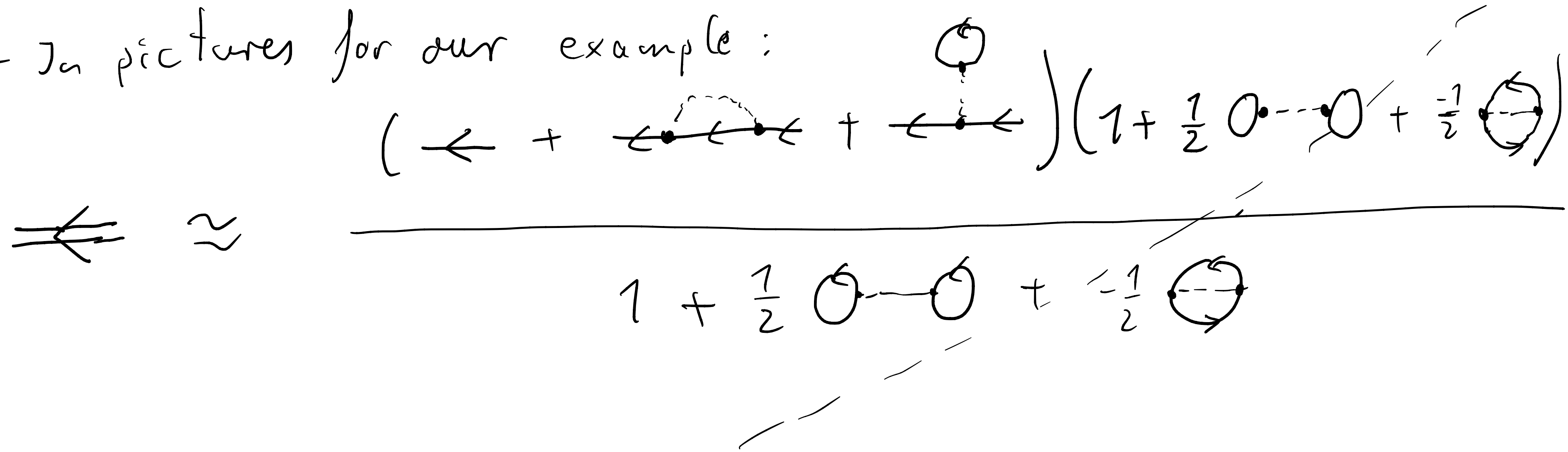
(ii) Graphical rep. of $-\frac{i}{2}\int d1 \int d2\, U(1-2) \langle 0 | T(\psi_1^+ \psi_2^+ \psi_2 \psi_1) | 0 \rangle$



- Crucial observation: Disconnected diagrams in numerator (i) contain identical components to denominator (ii).

• $\boxed{\text{Cancellation theorem: Vacuum diagrams in the denominator of } G_c \text{ exactly cancel all disconnected diagrams in the numerator of } G_c.}$

- In pictures for our example:



- General proof: Consider ν-th order in $W(t)$ term in the numerator:

$$\langle 0|T\left(e^{-i\int_{-\infty}^{+\infty}d\tau\, W(\tau)}\psi_r\psi_{r'}^\dagger\right)|0\rangle^{(\nu)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\delta_{m+n,\nu}\,(-i)^\nu\frac{1}{\nu!}\left(\frac{\nu!}{m!\,n!}\right)\times$$

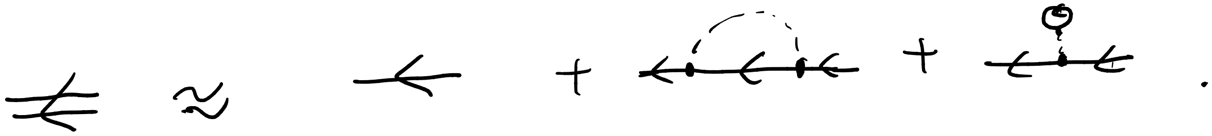
$$\times\int_{-\infty}^{+\infty}dt_1\dots dt_m\,\langle 0|T\left(W(t_1)\dots W(t_m)\psi_r\psi_{r'}^\dagger|0\rangle_{\text{conn.}}\right.\int dt_{m+1}\dots dt_\nu\,\langle 0|T\left(W(t_{m+1})\dots W(t_\nu)\right)|0\rangle$$

only connected contributions

$$\xrightarrow{\sum_{\nu=0}^{\infty}\dots}\ \langle 0|T\left(e^{-i\int_{-\infty}^{+\infty}d\tau\, W(\tau)}\psi_r\psi_{r'}^\dagger\right)|0\rangle = \left[\left(\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\infty}^{+\infty}dt_1\dots dt_m\,\langle 0|T\left(W(t_1)\dots W(t_m)\psi_r\psi_{r'}^\dagger\right)|0\rangle_{\text{conn}}\right)\right.$$

$$\times\underbrace{\left(\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{+\infty}dt_1\dots dt_n\,\langle 0|T\left(W(t_1)\dots W(t_n)\right)|0\rangle\right)}_{=\langle 0|e^{-i\int_{-\infty}^{+\infty}d\tau\, W(\tau)}|0\rangle,\ \text{cancelled by denominator.}}\ \square$$

$\rightsquigarrow$ The causal GF is the sum of the <u>connected</u> diagrams.



$+\mathcal{O}(u^2)$

- Diagrammatic rules

(1) Draw all topologically distinct Feynman diagrams

(2) Translate them into expressions with free causal GFs (all internal variables $1, \ldots, \ell$ are integrated over)

(3) Multiply every diagram by $(-1)^F$, where F is the number of closed propagator loops with more than one vertex "•" (i.e. free densities $\langle n \rangle_0 \sim$ ○• do not count, and neither do interaction lines)