- Linear response theory (see Problem sheet 1 for basic derivations)

- For $\tilde{H} = H + BF(t)$, we have for the linear response

[operator] [external field]

of a generic observable A:

Bosonic

$$\delta \langle A \rangle (t) = \int_{-\infty}^{+\infty} dt' F(t')\, G^r_{AB}(t,t'), \text{ with } G^r_{AB}(t,t') = -i\theta(t-t')\langle [A(t), B(t')] \rangle$$

where the expectation value and time-evolution is taken with respect to H, i.e. the Hamiltonian without the time-dependent perturbation, but with many-body interaction!

- Kubo-Martin-Schwinger identity:

$$\langle A(t) B(0) \rangle = \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H} e^{iHt} A e^{-iHt} B\right) = \langle B(0) A(t+i\beta) \rangle$$

[insert $e^{+\beta H} e^{-\beta H}$]

$$\hookrightarrow \; K_A(\omega) := \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\, \frac{1}{2}\langle \{A(t), A(0)\}\rangle \overset{t \to t+i\beta}{=} \frac{1}{2}\left(1+e^{\beta\omega}\right)\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\langle A(0)\, A(t)\rangle$$

$$\hookrightarrow \; \tilde{K}_A(\omega) := \int_{-\infty}^{+\infty} dt\, e^{i\omega t}\langle [A(t), A(0)]\rangle = \left(e^{\beta\omega} - 1\right)\int_{-\infty}^{+\infty} dt\, e^{i\omega t}\langle A(0)\, A(t)\rangle$$

$$\rightsquigarrow \quad K_A(\omega) = \frac{1}{2}\coth\left(\frac{\beta\omega}{2}\right)\tilde{K}_A(\omega).$$

Equilibrium

$$\langle [A(-t), A(0)]\rangle = \langle [A(0), A(t)]\rangle$$

Furthermore, we can rewrite $\tilde{K}_A(\omega) = \int_0^{\infty} dt\, e^{i\omega t}\langle [A(t), A(0)]\rangle$

$$- \int_0^{\infty} dt\, e^{-i\omega t}\langle [A(t), A(0)]\rangle = 2\,\mathrm{Re}\left(\int_0^{\infty} dt\, e^{i\omega t}\langle [A(t), A(0)]\rangle\right)$$

$A = A^{+}$ observable

Def. G^r

$$= -2\,\mathrm{Im}\left(G^r_{AA}(\omega)\right).$$

$\rightsquigarrow$ Fluctuation - dissipation theorem:

$$\boxed{K_A(\omega) = -\coth\left(\tfrac{\beta\omega}{2}\right) \operatorname{Im}\left(G^r_{AA}(\omega)\right)}$$

Fluctuations measured by autocorrelation function (noise power spectrum)

Dissipation determined by imaginary part of susceptibility.

- For fermions, observables A contain an even number of field operators (and are thus still bosonic), so that the $G^r_{AB}(t,t')$ are at least <u>two</u>-particle GFs (see A.1.d).

Important exception: The current in a fermionic system can be written in terms of the <u>single</u>-particle GF as:

$$j(x,t) = \frac{ie}{2m} \lim_{t' \to t+0^-} \lim_{x' \to x} \left(\nabla_{x'} - \nabla_x\right)\left(\overset{\text{fermions}}{\mp}\, G_c(x',t';x,t)\right).$$

For bosons, important susceptibilities, e.g. the dynamical susceptibility to lattice displacements (phonons) is directly determined by the phononic single-particle GF.

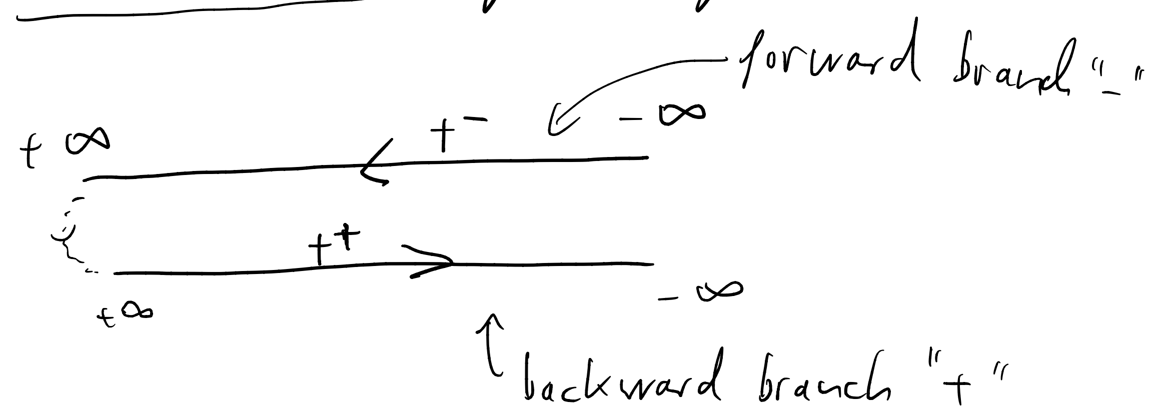
A.3.a Basic Schwinger–Keldysh formalism

- Key issue in out of equilibrium situations:

  $|\phi(\infty)\rangle = U^D(\infty,-\infty)|0\rangle$ is not naturally related to the initial state $|0\rangle$. $\rightsquigarrow$ Adiabatic switching procedure fails!

  ↳ $|\phi(\infty)\rangle$ needs to be explicitly evolved backwards to $t=-\infty$ by $U^{D\dagger}(\infty,-\infty)$.

  ↳ Time-ordering along the axis $+\infty \longleftarrow -\infty$ will be replaced by <u>contour-ordering</u> along the Keldysh contour γ:

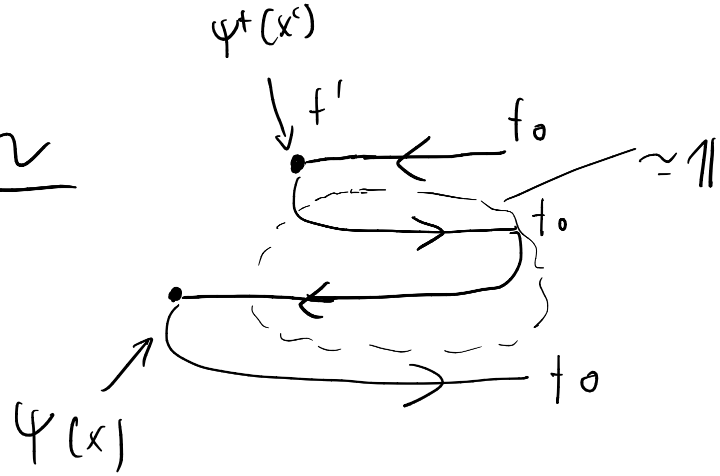


Contour-ordering operator T_K analogous to T, but ordering along the contour γ.

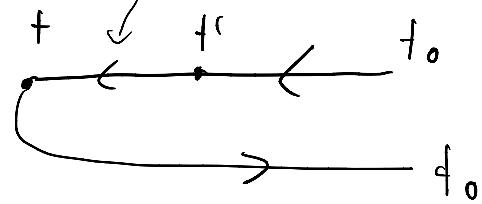
↳ Motivation from correlation functions in the Heisenberg picture

$$\langle T(\Psi_H(x,t)\Psi_H^\dagger(x',t'))\rangle \overset{\text{assume } t>t'}{=} \langle U^\dagger(t,t_0)\Psi(x)\underbrace{U(t,t_0)\,U^\dagger(t',t_0)}_{=U(t,t')}\Psi^\dagger(x')U(t',t_0)\rangle$$

$\simeq$



$\simeq$



$\boxed{U^\dagger(\infty,t)U(t,\infty)=\mathbb{1}}$

$\simeq$

$\boxed{t_0 \to -\infty}$