- Natural extension of causal GF to the Keldysh contour:

↳ $G_c(x_1, t_1; x_2, t_2) \equiv G^{--}(x_1, t_1; x_2, t_2) \equiv G_\gamma(x, t_1^-; x_2, t_2^-) \equiv$

$$\left\langle T_K\left(e^{-i\int_\gamma d\tau\, V_\tau^0(\tau)}\, \Psi(x_1, t_1^-)\, \Psi^+(x_2, t_2^-) \right) \right\rangle_0 ,$$

where $\langle \ldots\ldots \rangle_0 = \frac{1}{Z_0} \mathrm{Tr}\left(e^{-\beta H_0} \ldots \right)$ is an equilibrium expectation value at $t \to -\infty$.

↳ Initial correlations neglected by choosing H_0 as the non-interacting part of H.

↳ In the perturbative expansion, terms integrated along all of γ occur, and hence the time arguments of G_γ

must be considered in arbitrary combinations of branches.

$$\hookrightarrow G_\gamma(x_1,\tau_1;x_2,\tau_2) \equiv \left\langle J_K\left(e^{-i\int_\gamma d\tilde{\tau}\, V_\tau^D(\tilde{\tau})}\,\psi(x_1,\tau_1)\,\psi^\dagger(x_2,\tau_2)\right)\right\rangle_0$$

$$\simeq \begin{pmatrix} G^{--}(x_1,t_1;x_2,t_2) & G^{-+}(x_1,t_1;x_2,t_2) \\ G^{+-}(x_1,t_1;x_2,t_2) & G^{++}(x_1,t_1;x_2,t_2) \end{pmatrix}$$

$$\equiv \hat{G}(x_1,t_1;x_2,t_2).$$

Here, the 2x2 matrix structure of $\hat{G}$ results from breaking up the contour γ into its branches $+,-$ and t_i, $i=1,2$ are the real values of the abstract contour time variables τ_i.

$\hookrightarrow$ Back to the standard Heisenberg picture, the three new GFs are defined as

$$G^{-+}(x_1,t_1;x_2,t_2) = \overset{\text{Fermi}}{\pm} i\langle \Psi_H^+(x_2,t_2)\Psi_H(x_1,t_1)\rangle ;$$

$$G^{+-}(x_1,t_1;x_2,t_2) = -i\langle \Psi_H(x_1,t_1)\Psi_H^+(x_2,t_2)\rangle ;$$

$$G^{++}(x_1,t_1;x_2,t_2) = -i\langle \tilde{T}\left(\Psi_H(x_1,t_1)\Psi_H^+(x_2,t_2)\right)\rangle ;$$

anti-time ordering from larger to smaller times

familiar causal GF

$$G^{--}(x_1,t_1;x_2,t_2) = -i\langle T\left(\Psi_H(x_1,t_1)\Psi_H^+(x_2,t_2)\right)\rangle = G_c(x_1,t_1;x_2,t_2).$$

- perturbative expansion of the contour-ordered GF

$$iG_\gamma(r,r') = \sum_{\nu=0}^{\infty}\frac{1}{\nu!}(-i)^\nu \int_\gamma d\tau_1 \dots \int_\gamma d\tau_\nu \left\langle T_K\left(V^0_{\tau_1}(\tau_1)\dots V^0_{\tau_\nu}(\tau_\nu)\,\Psi(r)\,\Psi^+(r')\right)\right\rangle_0$$

- Lowest order of perturbative expansion for G_γ with interaction $\frac{1}{2}\int dx_1\,dx_2\,U(x_1-x_2)\,\Psi^+_{x_1}\Psi^+_{x_2}\Psi_{x_2}\Psi_{x_1}$:

$$i G_\gamma(r,r') \underset{\uparrow \atop +\mathcal{O}(U^2)}{\approx} \langle T_K(\psi_r \psi_r^+)\rangle_0 + \frac{-i}{2}\int_\gamma d\tau_1 \int_\gamma d\tau_2 \int dx_1 \int dx_2 \langle T_K(\psi_1^+\psi_2^+\psi_2\psi_1\psi_r\psi_{r'}^+)\rangle_0 \times \delta(\tau_1-\tau_2)\, U(x_1-x_2).$$

$$= G_\gamma^0(r,r') + \frac{-i}{2}\int d1 \int d2\, U(1-2) \langle T_K(\psi_1^+\psi_2^+\psi_2\psi_1\psi_r\psi_{r'}^+)\rangle_0$$

$$\langle T_K(\psi_1^+\psi_2^+\psi_2\psi_1\psi_r\psi_{r'}^+)\rangle_0 \overset{\text{wick}}{=} \underbrace{+i G_\gamma^0(1,1^+)\, i G_\gamma^0(2,2^+)\, i G_\gamma^0(r,r') - i G_\gamma^0(1,2)\, i G_\gamma^0(2,1)\, i G_\gamma^0(r,r')}_{\text{disconnected}}$$

$$+ i G_\gamma^0(r,1)\, i G_\gamma^0(1,2)\, i G_\gamma^0(2,r') + i G_\gamma^0(r,2)\, i G_\gamma^0(2,1)\, G_\gamma^0(1,r')$$

related by $1 \leftrightarrow 2$

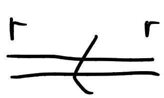
$$+ i G_\gamma^0(r,1)\, i G_\gamma^0(1,r') \underbrace{(-i G_\gamma^0(2,2^+))}_{n(2)} + i G_\gamma^0(r,2)\, i G_\gamma^0(2,r') \underbrace{(-i G_\gamma^0(1,1^+))}_{n(1)}$$

↳ Disconnected diagrams cancel out, since the forward and backward branch of γ give equal contributions but opposite sign.

↳ For the same reason, there is no explicit denominator of vacuum diagrams since $\left\langle J_K\left(e^{-i\int_\gamma d\tau\, V_\tau^0(\tau)}\right)\right\rangle_0 = \langle \underbrace{U_0^\dagger(\infty,-\infty)\, U_0(\infty,-\infty)}_{\mathbb{1}} \rangle_0$

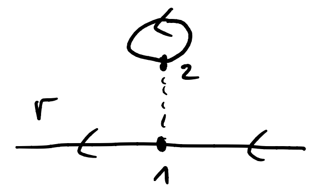
$= 1.$

↳ Formally analogous to the $T=0$ case, we thus have



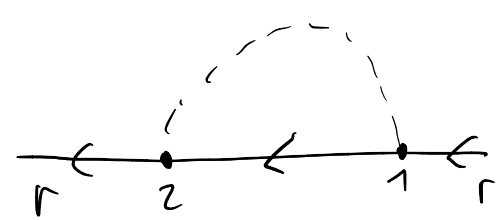
$\simeq$

$+$



$$iG(r,r') \simeq iG^0_\gamma(r,r') + \int d1 \int d2\, (-iU(1-2))\, iG^0_\gamma(r,1)\, iG^0_\gamma(1,r')\left(-i\, G^0_\gamma(2,2^+)\right)$$

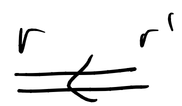
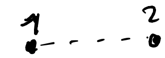
$+$



$$\int d1 \int d2\, (-iU(1-2))\, i\, G_\gamma(r,2)\, i\, G^0_\gamma(2,1)\, i\, G^0_\gamma(1,r')$$

with the translation table

(All internal variables are integrated over space and γ)

			
$iG_\gamma(r,r')$	$iG^0_\gamma(r,r')$	$-i\mathcal{U}(1-2)$	$-iG^0_\gamma(1,1^+) = n(1)$

- Real-time formalism:

$$\hookrightarrow \int_\gamma d\tau \ldots \simeq \int_{-\infty}^{+\infty} dt \ldots + \int_{+\infty}^{-\infty} dt \ldots = \sum_{\eta=\pm} \eta \int_{-\infty}^{+\infty} dt \ldots$$

$\hookrightarrow$ Contour-time delta-functions: $\displaystyle\int_\gamma d\tau \ldots = \int_\gamma d\tau_1 \int_\gamma d\tau_2\, \delta(\tau_1-\tau_2) \ldots =$

$$= \sum_{\eta_1,\eta_2} \eta_1 \int_{-\infty}^{+\infty} dt_1\, \eta_2 \int_{-\infty}^{+\infty} dt_2\, \delta_{\eta_1,\eta_2}\, \eta_1\, \delta(t_1-t_2) \ldots$$

$$\rightsquigarrow \int_\gamma d\tau_1 \int_\gamma d\tau_2 \int dx_1 \int dx_2 \left(-i\mathcal{U}(x_1-x_2)\right) \delta(\tau_1-\tau_2) = \sum_{\eta_1,\eta_2} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \int dx_1 \int dx_2 \left(-i\mathcal{U}(x_1-x_2)\right) \delta_{\eta_1,\eta_2}\, \eta_1\, \delta(t_1-t_2)$$