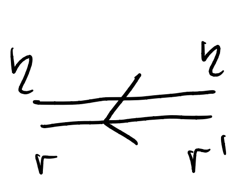
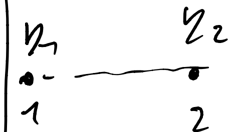
$$\hookrightarrow i\hat G(r,r')_{\eta\eta'} \overset{\mathcal{O}(U^2)}{\approx} i\hat G_0(r,r')_{\eta\eta'} + \sum_{\eta_1\eta_2} \underbrace{\int_{-\infty}^{+\infty}dt_1\int_{-\infty}^{+\infty}dt_2\int dx_1\int dx_2}_{\int d\hat 1\int d\hat 2} \big(-iU(x_1-x_2)\big)\,\delta_{\eta_1\eta_2}\,\eta_1 \times$$

$$\times\,\delta(t_1-t_2)\Big[\, i\hat G^0(r,\hat 1)_{\eta\eta_1}\; i\hat G^0(\hat 1,\hat 2)_{\eta_1\eta_2}\; i\hat G^0_{\eta_2\eta'}(\hat 2,r') \;+$$

$$+\; i\hat G^0(r,\hat 2)_{\eta\eta_2}\; i\hat G^0(\hat 2,r')_{\eta_2\eta'}\,(-i)\,\hat G^0_{\eta_1\eta_1}(\hat 1,\hat 1^+)\Big]$$

$\rightsquigarrow$ Decoding table:
(All internal variables and Keldysh indices η_i are integrated/summed over)

 $i\hat G_{\eta\eta'}(r,r')$	 $i\hat G^0_{\eta\eta'}(r,r')$	 $\underbrace{-iU(x_1-x_2)\,\eta_1\,\delta_{\eta_1\eta_2}\,\delta(t_1-t_2)}_{\equiv\, -i\hat U(\hat 1-\hat 2)\,=\, iU(\hat 1-\hat 2)\hat\tau_3}$
 $-i\hat G^0_{\eta_1\eta_1}(\hat 1,\hat 1^+) = n(1)$		

⇝ Main differences to equilibrium theory: Every vertex i has a Keldysh index η_i, where vertices connected by an interaction line have identical Keldysh indices (as they correspond to the same physical time).

[positive real time direction] [„earlier in J_K"]

<u>Warning</u>: $\eta = \pm$ corresponds to the $\mp$ branch of the contour γ

[negative real time direction] ["later in contour ordering"]

↳ Larkin-Ovchinnikov representation: Consider transformation in Keldysh space with $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ of $\hat{\tau}_3 \hat{G} \equiv \check{G}$,

i.e. $\check{G} = U \hat{\tau}_3 \hat{G} U^\dagger = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} G^{--} & G^{-+} \\ -G^{+-} & -G^{++} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} =$

$= \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}$, where $\overset{\text{retarded}}{G^R} = G^{--} - G^{-+} = G^{+-} - G^{++}$

$$G^A = G^{--} - G^{+-} = G^{-+} - G^{++} \quad (\text{advanced})$$

$$G^K = G^{\pm} + G^{\mp} = G^{--} + G^{++} \quad (\text{Keldysh GF})$$

Explicitly $$G^K(x,t\,;\,x',t') = -i\Big\langle \big[\psi_H(x,t),\, \psi_H^{+}(x',t')\big]_{\underset{\text{Bose}}{\overset{\text{Fermi}}{\mp}}} \Big\rangle$$

$$G^{--} + G^{++} \overset{\text{Fermi}}{=} -i\langle T(\psi_r \psi_{r'}^{+})\rangle + -i\langle \tilde{T}(\psi_r \psi_{r'}^{+})\rangle$$

$$= -i\langle [\psi_r , \psi_{r'}^{+}]\rangle \checkmark$$

either T or $\tilde{T}$ swaps operators and gives extra "−"

$$G^{--} - G^{-+} \overset{\text{Fermi}}{=} -i\,\theta(t-t')\langle \psi_r \psi_{r'}^{+}\rangle + \underbrace{i\,\theta(t'-t)\langle \psi_{r'}^{+} \psi_r\rangle - i\langle \psi_{r'}^{+} \psi_r\rangle}$$

$$= -i\,\theta(t-t')\langle \psi_r \psi_{r'}^{+}\rangle - i\,\theta(t-t')\langle \psi_{r'}^{+}\psi_r\rangle = -i\theta(t-t')\langle\{\psi_r , \psi_{r'}^{+}\}\rangle$$

$$= G^R(r,r') \checkmark$$

- Keldysh perturbation theory for (time-dependent) external potential $V(x,t)$.

$\uparrow$ single-particle

↳ Dyson equations for G_γ replaced by:

$$G_\gamma(1,1') = \int d2\, G^0_\gamma(1,2)\, V(2)\, G_\gamma(2,1') + G^0_\gamma(1,1')$$

$$= \int d2\, G_\gamma(1,2)\, V(2)\, G^0_\gamma(2,1') + G^0_\gamma(1,1')$$

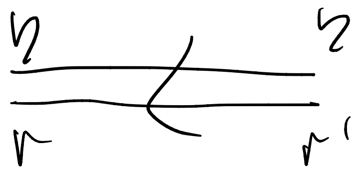
↳ Real-time formalism $\quad \int_\gamma d\tau \to \sum_{\eta=\pm} \eta \int_{-\infty}^{+\infty} dt \quad$ with $\hat{V}(1) = V(1)\,\hat{C}_3$

$$\boxed{\hat{G} = \hat{G}^0 + \hat{G}^0 \hat{V} \hat{G} = \hat{G}^0 + \hat{G} \hat{V} \hat{G}^0}$$

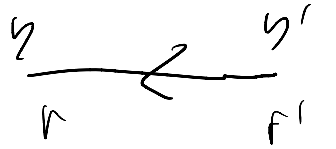
(integration on internal space-time variables implied)

↳ Diagrammatic representation:

$i\hat{G}_{\eta\eta'}(r,r')$



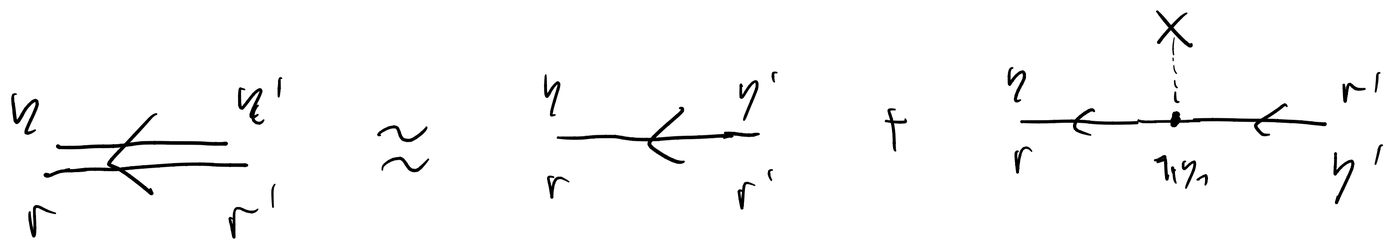
$i\hat{G}^{0}_{\eta,\eta'}(r,r')$



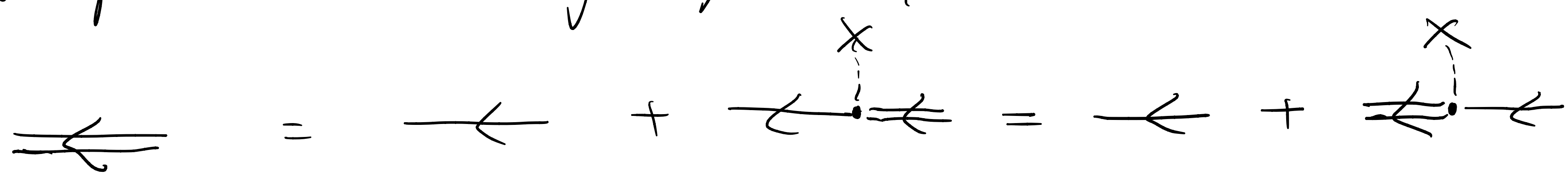
$-i\hat{V}_{\eta\eta'}(1) = \frac{V(1)}{i}\eta\delta_{\eta,\eta'} = \frac{V(1)}{i}\hat{\tau}_3^{\eta\eta'}$

$\boxed{\hat{\tau}_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$

↳ First-order diagram:



↳ Diagrammatic version of Dyson equations:



↳ Scattering potential $V(1)$ plays the role of the self-energy, but has only one physical vertex and is always diagonal in Keldysh space

↳ No energy (momentum) conservation between external particle lines, if V depends on time (space).

A.3.b: Dyson- and Kadanoff-Baym equations

- Dyson equations for G_γ:

$$G_\gamma(1,1') = G^0_\gamma(1,1') + \int d2 \int d3\, G^0_\gamma(1,2)\, \Sigma_\gamma(2,3)\, G_\gamma(3,1') \qquad (D1)$$

$$= G^0_\gamma(1,1') + \int d2 \int d3\, G_\gamma(1,2)\, \Sigma_\gamma(2,3)\, G^0_\gamma(3,1') \qquad (D2)$$

↳ The self-energy is diagrammatically defined analogous to the $T=0$ equilibrium case, simply by replacing diagrams for G_c with their analogs entering G_γ.

↳ The two equations (D1), (D2) are not independent, since the convolution of the self-energy with GFs cannot be turned into a product of numbers in reciprocal space (as in equilibrium).

- Real time formalism:

↳ Breaking up contour integrals into real-time integrals yields:

$$\hat{G} = \hat{G}^0 + \hat{G}^0 \hat{\tau}_3 \hat{\Sigma} \hat{\tau}_3 \hat{G}$$

(integration on internal space-time variables implied)

With $\check{G} \equiv \hat{\tau}_3 \cdot \hat{G} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix}$, we get

$$\boxed{\begin{aligned} \check{G} &= \check{G}_0 + \check{G}_0 \check{\Sigma} \check{G} \\ &= \check{G}_0 + \check{G} \check{\Sigma} \check{G}_0 \end{aligned}}$$

(Keldysh-Dyson)

(Form-invariant under unitary basis transformations from $\check{G}$ to $\overline{G}$.